Shopping effort, variable capital utilization, and business cycle decomposition

Mario Rafael Silva, Marshall Urias

Abstract

We develop a two-sector business cycle model in which aggregate demand affects total factor productivity through variable capital utilization and shopping intensity. We estimate the model by Bayesian means on a rich set of observables including the standard and utilization-adjusted Solow residuals. We find that shopping demand shocks and technology shocks are equally important in explaining variation in output, the Solow residual, and investment. While technology shocks induce nearly perfect comovement between the standard and utilization-adjusted Solow residual, shopping demand shocks only increase the standard Solow residual. When the model is estimated without using the utilization-adjusted Solow residual, then technology shocks account for most of the variation in output and TFP, and the two productivity measures comove very closely. Hence, positive demand shocks can appear as technology enhancing if one uses the conventional measure of the Solow residual when in reality there is no such change.

Keywords: goods market frictions, firm entry, endogenous variety, endogenous productivity, Bayesian estimation

JEL Classification: D10; E21; E22; E32; E37

1. Introduction

A traditional view of business cycles, which came to prominence after the Great Depression, is that they are characterized not by a loss of productive ability but instead by insufficient employment of resources. John Maynard Keynes wrote in 1930 that the world was 'as capable as before of affording for every one a high standard of life...today, we have involved ourselves in a colossal muddle.' In line with this view, business cycles tend to be characterized by a reduction in utilization. Figure 1 plots the 4 quarter percent change in the

Email address: msilva913@hkbu.edu (Mario Rafael Silva)

Solow residual and the utilization-adjusted measure. In the second quarter of 2020, total factor productivity declined by nearly 17% at an annualized rate. Adjusted for utilization from the methodology in Basu, Fernald, and Kimball (2006), TFP actually grew by 1.46%. Over the past four quarters in 2021Q1, TFP grew at 2.76%, and utilization-adjusted TFP great at 0.15%. TFP also fell below its utilization-adjusted counterparted in previous recessions, including the Great Recession, though the difference was less dramatic.

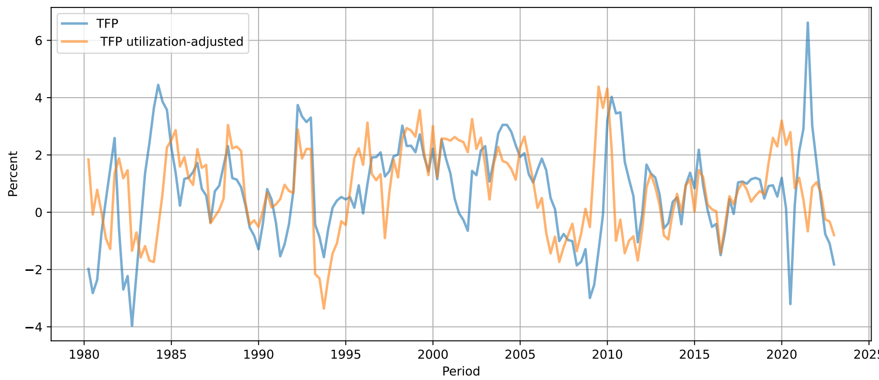

Figure 1: 4 Quarter Percent Change

More generally, utilization-adjusted TFP, which better proxies technology, has substantially different time series properties than the Solow residual. Two key features that stand out are that the utilization-adjusted measure is less volatile and less correlated with output. Table 1 shows that this feature generally holds across standard ways of decomposing the cycle and trend, with the partial exception of the linear-quadratic filter. The estimation under the growth filter is especially salient since it is a popular choice for constructing observable series used in estimation.[1]

[1] To be consistent with the Kalman filter, observable series used in estimation cannot be obtained using a filter which involves the use of future data. This rules out the standard HP filter and BK filter, though the one-sided HP filter is still admissible. The Hamilton regression filter can be used for both description and estimation.

	Hamilton		Quadratic		HP filter		BK filter		Growth	
	SR	SR_{util}	SR	SR_{util}	SR	SR_{util}	SR	SR_{util}	SR	SR_{util}
SD(x)	2.48	2.08	3.57	3.54	1.28	0.93	1.16	0.86	0.85	0.80
RSD	0.77	0.64	0.87	0.87	0.81	0.59	0.84	0.63	0.76	0.72
Cor(x, Y)	0.73	0.15	0.52	0.31	0.79	0.00	0.76	-0.05	0.80	0.10

Table 1: Business cycle statistics under different filters. The time range is 1960Q1-2022Q4. The Hamilton filter uses $h = 8$ quarters and lags $p = 4$. HP filter uses smoothing parameter $\lambda = 1600$, and BK filter uses cutoff frequencies 6 and 32.

Indeed, in their seminal analysis, Basu, Fernald, and Kimball (2006) find that technology fluctuates much less than TFP and that TFP may lag technology. Moreover, consumption Granger-causes TFP whereas TFP does not Granger-cause consumption. Together, these fact suggests that TFP depends on utilization of inputs and matching firms and consumers, not just technology.

A long-standing problem in macroeconomics, motivated early on by Keynes' *General Theory* and exemplified by modern DSGE models like Smets and Wouters (2007), is to better understand the sources of business cycle fluctuations. In particular, how can such cycles be decomposed into demand, supply, and other factors? As the Solow residual depends on technology and demand and is mediated by frictions, it is a prime observable series to use for estimation. The purified measure of TFP, additionally, provides much more additional information by stripping out the role of input utilization. Effectively, the model must try to fit two very different productivity time series.

Accordingly, we develop a dynamic stochastic general equilibrium model in which search frictions, imperfect competition, and variable capital utilization give rise to a productive role of demand in both the consumption and investment sectors. These ingredients help us delineate the extent to which goods market frictions matter alongside firms' decisions to utilize capital more intensively.

In the model, two imperfectly competitive sectors sell differentiated goods subject to search frictions: a consumption (retail) sector and investment sector. Households shop for consumption goods, supply labor, and save through equity shares of firms; capital goods suppliers employ labor to shop for investment goods and transform them into capital; and the government purchases consumption goods alongside households.

The model incorporates the use of both variable and fixed factors of production. Production in both the consumption and investment sectors requires some overhead labor and capital. As Huo and Ríos-Rull (2018), capital is fixed in each location, whereas variable labor can be dispatched to satisfy orders once customers arrive. Similarly, capital goods suppliers employs variable labor used to shop for investment goods which are subsequently transformed into capital. Accordingly, firms can adjust their productive capacity through employment of overhead labor and capital, while also allowing short-run utilization adjustments through dispatching variable labor and capital intensity. The elasticity of production with respect to variable versus fixed factors will determine how important demand shocks are in explaining variation in output. Changes to productive capacity are subject to investment adjustment costs, which introduces a wedge between the cost of increasing acquiring new capital and the value it brings to the firm, resulting in smoothing adjustments of the capital stock in response to shocks.

Households have preferences of the type studied by Jaimovich and Rebelo (2009) type over consumption and labor supply, augmented with disutility over shopping effort. These preferences provide flexibility in the estimation to determine the strength of short-run wealth effects on labor supply. We find that the model requires very little wealth effects to match the data.

These features contrast with a standard real business cycle model, in which output is a function of inputs and prices adjust so that all produced goods are consumed. While a New Keynesian model provides a role for demand shocks to affect output due to price stickiness, the mechanism is otherwise different. Whereas the (intratemporal) elasticity of substitution and markup only affect dynamics through their interaction with nominal rigidities, here it is a fundamental determinant of shopping for consumption and investment goods and, in the extended model, setting up new product lines.

The most notable paper studying goods market frictions in a business cycle setting is Bai, Rios-Rull, and Storesletten (2012). They also utilize a two-sector framework and consider exogenous disturbances to preferences and productivity. As households cut back on shopping, some fraction of capital and labor is unmatched, which lowers utilization and the Solow residual. The authors estimate the model by Bayesian means and find that preference shocks explain most of the variation in output and the Solow residual. Additionally, demand shocks and search frictions help explain variation in capacity utilization.

There are several major innovations with respect to Bai, Rios-Rull, and Storesletten (2012). First, we include a much wider array of observable series in estimation, most notably the utilization-adjusted Solow and the relative price of investment. Second, we allow for both technology and demand shocks to be correlated in the consumption and investment structures. Third, we incorporate variable capital utilization: the use of capital depends on both successful matching with consumers and intensity subject to higher depreciation. Fourth, we also incorporate government purchases of consumption goods and endow government with a shopping technology. Firms specialize in either servicing consumers or the government, but they are free to change their decision each period. Finally, in the extended model, we incorporate endogenous firm entry, which Bilbiie, Ghironi, and Melitz (2012) and subsequent papers have shown help match second moments of labor hours and investment. Additionally, it provides us an additional margin of overhead labor and capital.

To understand how the responses depend on the type of disturbance, first consider a positive preference shock. Consumers spend more on existing varieties and shop more to expand their basket of goods. Increased demand for labor pushes up wages and labor-intensive entry costs. Productivity and firm revenue also rise due to improved matching efficiency. Provided the shock is sufficiently persistent, the discounted value of firm profits rises enough to promote entry. Aggregate firm profits increase from both higher sales of incumbents and entry. In the absence of shopping effort, consumption variety expands by less and there is no effect on firm productivity.

We estimate the baseline model by Bayesian means to data on nine series: output, consumption, investment, labor supply, real wages, government consumption expenditures, the relative price of investment, and the adjusted and unadjusted Solow residuals. For the extended model, we also use data on firm entry. These are the real series used by Lewis and Stevens (2015), plus both Solow residuals and the relative price of investment.[2] It also is a

[2]Lewis and Stevens (2015) study a monetary model and thus also include data on interest rates and inflation. Moreover, Offick and Winkler (2019) also make use of aggregate profits. In general, a two-sector model of the vein of Bilbiie, Ghironi, and Melitz (2012) cannot match the volatility of profits relative to

superset of the variables used by Bai, Rios-Rull, and Storesletten (2012). The choice of these series is intuitively reasonable. Consumption and output/investment are important series for disentangling demand and technology shocks, as is the relative price of investment to consumption. Demand shocks typically induce proportionately larger effects on consumption relative to investment. The relative price of investment also helps disentangle different type of demand and technology shocks. Data on investment directly disciplines the size of both product creation and development of physical capital. The two types of Solow residuals are also crucial in determining the extent of movements in technology and utilization.

There are stochastic disturbances to (intratemporal) consumption preferences, the discount factor, consumption, shopping disutility, investment shopping efficiency, sector-specific technology shocks, and labor supply (disutility) shocks. Each shock, other than that of labor supply, follows an AR(1) process in logs. We specify labor supply as an AR(2) process to capture both higher and lower frequency movements, as by Bai, Rios-Rull, and Storesletten (2012). We also include measurement errors in wages, investment, and labor supply.

We find that demand shocks–especially those related to shopping–play an important role in explaining the variation of output, the Solow residual, and, to a lesser extent, investment within a 2-year horizon. Technology shocks play a more important role at longer horizons. Labor supply shocks also drive substantial variation in output and investment but do not matter for the Solow residual. Whereas all demand shocks stimulate shopping, labor supply, and the Solow residual, intratemporal preference shocks tend to slightly lower the relative price of investment. Finally, if the model is estimated without using the utilization-adjusted Solow residual, then technology shocks explain a majority of variation of output and the Solow residual even in short horizons. Moreover, both types of TFP are extremely correlated with each other and similarly correlated with output, contrary to the data.

1.1. Related literature

Exploring endogenous sources of Solow residual fluctuations is related to the capacity utilization literature. Early work includes Greenwood, Hercowitz, and Huffman (1988) and Basu (1996) among others. Shapiro (1993), for instance, finds that much of the cyclicality of TFP can be attributed to capital's workweek, which is consistent with Table 1.

output, so measurement error ends up absorbing much of the variability in profits.

We follow the practice in many estimated DSGE models of allowing for variable capital utilization, but we also incorporate symmetric goods market frictions in the consumption and investment sectors as in Bai, Rios-Rull, and Storesletten (2012). However, we model them using endogenous variety and random search as in Huo and Ríos-Rull (2018). We believe this approach most naturally incorporates search frictions in an imperfectly competitive setting common to modern medium-scale DSGE models, and in the extended version also relates to endogenous-entry models in the vein of Bilbiie, Ghironi, and Melitz (2012).

This paper connects closely with Huo, Levchenko, and Pandalai-Nayar (2023). They develop a multi-sector multi-country model which yields an estimating equation that allows one to adjust the Solow residual for utilization. The approach is very similar to Basu, Fernald, and Kimball (2006) except for its multi-sector open-economy nature and explicit modeling of the household sector. They find that the utilization-adjusted TFP is virtually uncorrelated across countries even though the Solow residual has a moderate correlation. They use the model to extract a utilization shock that captures the effects of all non-TFP shocks on utilization rates. They find that TFP shocks account for very little GDP comovement whereas utilization shocks generate 1/3 of the observed comovement. Our work, though restricted to a domestic framework, can be understood as decomposing utilization into shopping frictions and variable capital utilization and explicitly accounting for a variety of demand shocks.

In addition to the measure of utilization we use based on Fernald (2014), there are survey-based direct measures of plant capacity utilization (i.e. Gorodnichenko and Shapiro (2011)) or electricity consumption (i.e. Burnside, Eichenbaum, and Rebelo (1995)). While these direct measures are correlated with the indirect measure by Fernald (2014), they are not appropriate to use as an economy-wide observable series for Bayesian estimation.

The model nests a two-sector real business-cycle model, as by Boldrin, Christiano, and Fisher (2001). Whereas they utilize imperfect intersectoral factor mobility and habit formation to address asset-pricing puzzles in the one-sector model, our primary motivation is to incorporate goods market frictions specific to the consumption sector. Additionally, we wish to allow for sector-specific technology shocks.

We incorporate three core features used by Jaimovich and Rebelo (2009) in studying both contemporaneous and news shocks in a two-sector real business cycle model. These consist of variable capital utilization, investment adjustment costs, and preferences which parameterize short-run wealth effects on labor supply. Taken together, firms have an incentive to smooth investment, and in the short run can satisfy greater production requirements by boosting utilization. The rise in labor supply depends on the strength of short-run wealth effects. This setup is thus more general than Bai, Rios-Rull, and Storesletten (2012), which uses additively separable preferences, and Huo and Ríos-Rull (2018), which assumes GHH preferences that altogether eliminate short-run wealth effects.

Finally, the paper is inspired by Smets and Wouters (2007), who estimate an enriched New Keynesian model on output, labor supply, consumption, investment, wages, inflation, and the nominal interest rate. We include same real series and also incorporate together data on government consumption, business formation, and both the unadjusted and adjusted Solow residuals. Several shocks correspond closely: labor supply shocks play a similar role as the wage markup shocks, and discount-factor shocks are related to the risk-premium shocks. However, our framework provides a rich role for goods market frictions and endogenous utilization. There are correlated demand shocks involving shopping as well as correlated sector-specific technology shocks.

The structure of the paper is as follows. Sections 2 and 3 lay out the environment and equilibrium. Section 4 discusses the quantitative results, and Section 5 concludes. The appendices describe the data sources, derive and list equilibrium conditions, and provide additional results from the estimation.

2. Environment

2.1. Quantity and price indices

There are two imperfectly competitive sectors, consumption and investment, where firms produce horizontally differentiated varieties of goods. The consumption c and investment i bundles take the form of Dixit-Stiglitz aggregators:

$$c_t = \left(\int_0^{\mathcal{A}^{\mathcal{C}}} c_{j,t}^{(\varepsilon-1)/\varepsilon} di \right)^{(\varepsilon/(\varepsilon-1))} \tag{1}$$

$$i_t = \left(\int_0^{\mathcal{A}^{\mathcal{I}}} i_{j,t}^{(\varepsilon_i-1)/\varepsilon_i} di \right)^{(\varepsilon_i/(\varepsilon_i-1))} \tag{2}$$

where $\varepsilon, \varepsilon_i$ are the elasticity of substitution parameters and $\mathcal{A}_C, \mathcal{A}_I$ are measures of the potential set of goods. For simplicity, we assume the same elasticity of substitution across sectors. Associated with (1) and (2) are the following welfare-based price indices:

$$P_t^C = \left(\int_{j \in \mathcal{A}_t^C} (p_{j,t}^c)^{1-\varepsilon} di \right)^{1/(1-\varepsilon)}$$

$$P_t^I = \left(\int_{j \in \mathcal{A}_t^I} (p_{j,t}^i)^{1-\varepsilon_i} dj \right)^{1/(1-\varepsilon_i)}$$

where a retail firm j posts price $p_{j,t}^c$ and an investment-goods firm posts price p_{jt}^i. The quantity $P_t c_t = \int_{j \in \mathcal{A}_t^c} p_{j,t} c_{j,t} dj$ is the minimal cost of obtaining one unit of consumption, and an analogous expression holds for the investment sector. Under a symmetric equilibrium, $1/(1-\varepsilon)$ is the elasticity of the price index with respect to the range of goods and thus measures the love of variety. It is also useful to define the relative prices $\rho_{j,t}^C = p_{j,t}^c / P_t^C$ and $\rho_{j,t}^I = p_{j,t}^i / P_t^I$. Higher consumption diversity lowers the price indices and therefore raises the relative prices.

2.2. Matching and production technology

There is a measure N_t of firms. Each period, firms can choose whether to produce consumption goods or investment goods. Within the former, retailers can further specialize in selling to households or government. There are thus three matching submarkets: consumers and retailers, government and retailers, and between the firms and capital goods suppliers for the investment good.

Each firm owns a measure 1 of locations, which may be matched with households or not. The production function at each location is

$$F^i(k, l_1, l_2) = Z^i (uk)^\alpha l_1^{\alpha_1} l_2^{\alpha_2}$$

where $\alpha_1 + \alpha_2 = 1 - \alpha$ and u is the capital utilization rate. Here, l_1 represents fixed labor at each location and l_2 represents variable labor that can be dispatched to a location to satisfy customer orders. Firms can increase utilization at the cost of higher depreciation, which is allowed to vary by sector, according to the functions $\delta^C(\cdot)$ and $\delta^I(\cdot)$ which are increasing and convex. We assume the functional form

$$\delta^i(u_t^i) = \delta^K + \beta_1^i (u_t^i - 1) + \frac{\beta_2^i}{2} (u_t^i - 1)^2, \quad \text{i} \in \{C, I\}$$

where δ^K is an exogenous rate of depreciation. We will restrict the parameterization, so that in the steady state utilization equals unity in each sector, and thus δ^K is the economy-wide steady-state depreciation rate of capital.

Similar to Huo and Ríos-Rull (2018), there is a constant returns to scale matching function between shopping effort and the measure of firms. If a shopper matches with a firm, then it is randomly allocated to one of its locations. The aggregate number of shopper-firm matches in submarket i is $M_t^i \subset \mathcal{A}^i$ given by

$$M_t^i = A(S_t^i)^{\phi}(N_t^i)^{1-\phi}$$

Consider the submarket H where consumers and retailers meet: the quantity S^H represents aggregate household shopping effort

$$S_t^H = \int_0^1 s_{i,t} di \tag{3}$$

for individual search units s_{it}. In the case of the submarket where the government shops for retail goods G, we assume that there is a consolidated shopping entity so there is no need to sum as in (3). Finally, in the submarket for investment goods I, one unit of labor provides ζ shopping units. Total shopping effort is undertaken by a unit mass of capital goods suppliers and is thus

$$S_t^I = \int_0^{N_t} \zeta l_{it}^K di$$

Define market tightness in submarket i as the ratio of firms per unit of shopping effort: $Q_t^i = N_t^i / S_t^i$. The measure of matches for a particular firm is $\mu_N(Q_t) = M_t/N_t = A(Q_t^i)^{-\phi}$ and the measure of matches for a single unit of search effort is $\mu_S(Q_t) = M_t/S_t = A(Q_t^i)^{1-\phi}$. In the household sector, the matches of an individual shopper with $s_{j,t}$ search units are $s_{j,t}A(Q_t^H)^{1-\phi}$. Note that, upon aggregating, $\int_0^1 s_{i,t}A(Q_t^H)^{1-\phi}di = S_t^H A(Q_t^H)^{1-\phi} = M_t^H$. Thus, the amount of product variety of either consumption or investment goods depends on shopping effort and market tightness. Implicitly, shopping allows consumers and firms to lower the price index.

2.3. Capital goods suppliers

Capital is created by specialized firms who transform investment goods into capital, and then differentiates it into sector-specific capital goods. Capital goods suppliers employ labor to purchase the investment goods, but this market subject to shopping frictions as described above. These firms earn revenues by renting their capital at rate $r_i, i \in C, I$ and incur costs from paying labor and purchasing differentiated investment goods. The detailed problem of these agents is described in Section 3.2.

2.4. Households

We adopt Jaimovich and Rebelo (2009) preferences between consumption and labor supply and include costly shopping effort:

$$u(c, X, s, L) = \frac{1}{1-\sigma}\theta\left(c - \frac{\chi L^{\psi}}{\psi}X\right)^{1-\sigma} - \kappa s$$

where κ and γ are level and elasticity parameters of shopping; and X is the geometric average of current and past consumption,

$$X = c^{\gamma} X_{-1}^{1-\gamma}$$

making preferences over consumption and hours non-separable over time. Setting $\gamma = 1$ yields preferences of the class discussed by King and Rebelo (1999), and setting $\gamma = 0$ yields preferences proposed by Greenwood, Hercowitz, and Huffman (1988).

The budget constraint, in units of the investment good, is

$$P_t^H c_t + \nu_t x_{t+1} + T_t = w_t L_t + (\nu_t + d_t)x_t \tag{4}$$

The left-hand side of (4) describes the expenses: households consume a basket of goods c_t at price P_t^H, purchase equity shares x_{t+1} at price ν_t, and pay lump-sum taxes T_t. The right-hand side describes the sources of income: households earn wages $w_t L_t$, dividends $d_t x_t$, and have claims on current equity $\nu_t x_t$.

2.5. Government

The government consists of two separate departments. A fiscal authority chooses government expenditures each period according an AR(1) process: $G_t = \rho_g G_{t-1} + \varepsilon_t^g$, where ε_t^G is i.i.d. and normal. A procurement department allocates this expenditure among consumption goods and chooses shopping effort. The government's period utility function over consumption and shopping effort is

$$u^g(g, s) = g - \kappa s.$$

Each period, the government consumes a subset of goods of measure $S_t^g A(Q_t^g)^{1-\phi}$. Analogously to the consumer, the government has a price P_t^g which satisfies $P_t^g g_t = \int_{i \in \mathcal{A}_t^g} p_{it} g_{it}$. Though we assume the government pays the same prices as consumers for individual goods, the price index generally differs in tandem with the measure of varieties.

Figure 2 summarizes the role of multiple sectors and search frictions in the environment.

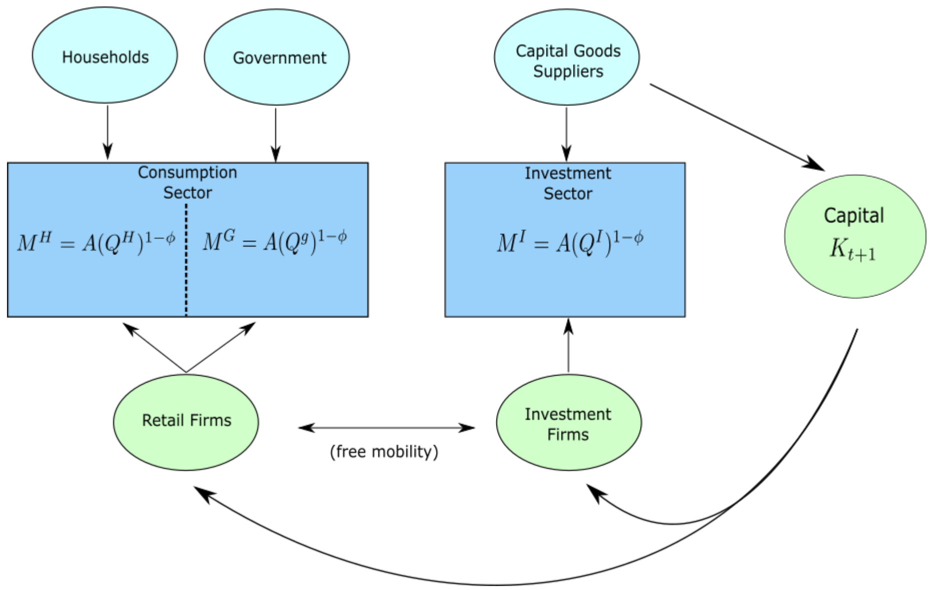

Figure 2: Role of multiple sectors and search frictions.

3. Equilibrium

3.1. Household problem

Consumers choose consumption c_t, shopping effort s_t, shares of equity x_t, and work hours L_t to maximize utility, given the laws of motion for firms and capital specified below. For convenience, we let the composite $\Gamma_t = c_t - \frac{\chi_t L_t^\psi}{\psi} X_t$ and note that agents internalize the dynamics of X_t. The relationship

$$P_t^H c_t = \int_0^{s_t A(Q_t^H)^{1-\phi}} p_{jt}^c c_{jt} dj \tag{5}$$

implicitly defines the consumption basket as a function of search effort s_t and individual consumption c_{it}. Given (5), the problem of the household is

$$\max_{c_{it}, L_t, s_t, x_t} \sum_{t=0}^{\infty} \beta^t b_t \left[\frac{1}{1-\sigma} \theta_t \left(c_t(c_{it}, s_t) - \chi_t \frac{L_t^\psi}{\psi} X_t \right)^{1-\sigma} - \kappa_t s_t \right]$$

subject to

$$P_t^H c_t(c_{it}, s_t) + \nu_t x_{t+1} + T_t = w_t L_t + (\nu_t + D_t) x_t \tag{6}$$

$$X_t = c_t^\gamma X_{t-1}^{1-\gamma} \tag{7}$$

Let λ_t be the Lagrangian multiplier on (6) and μ_t be the multiplier on (7). (Appendix B.1) solves the household problem in detail where the following relationships are obtained:

$$\lambda_t P_t^H = b_t \theta_t \Gamma_t^{-\sigma} - \mu_t \gamma \left(\frac{X_{t-1}}{c_t} \right)^{1-\gamma} \tag{8}$$

$$b_t \theta_t \Gamma_t^{-\sigma} \chi_t L_t^{\psi-1} X_t = \lambda_t w_t \tag{9}$$

$$b_t \kappa_t s_t = \frac{P_t^H \lambda_t c_t}{\varepsilon - 1} \tag{10}$$

$$\mu_t = \frac{b_t \theta_t \Gamma_t^{-\sigma} \chi_t L_t^{\psi}}{\psi} + \beta \mathbb{E}_t \left\{ \mu_{t+1} (1-\gamma) c_{t+1}^{\gamma} X_t^{-\gamma} \right\} \tag{11}$$

$$\nu_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} (\nu_{t+1} + D_{t+1}) \right\} \tag{12}$$

The Lagrangian multiplier in (8) summarizes the influences of the intertemporal shock b_t, intratemporal shock θ_t, risk aversion and habit formation, and the welfare-based price index P_t^h. Equation (9) is the standard labor supply condition. Equation (10) is novel and ties shopping effort to the level of the consumption basket and the shopping disutility. Equation (11) governs the optimal change of wealth effects over time. Equation (12) is the standard asset-pricing condition for equity: the value of a share equals future price and dividends adjusted by the intertemporal marginal rate of substitution.

Several insights can be obtained from equation (10). First, the value of shopping depends on an imperfect ability to substitute among goods, as $s \to 0$ as $\varepsilon \to \infty$. Second, the consumption level has opposing effects on shopping effort. Consumption expenditure is a prerequisite for shopping effort, but it also tends to reduce shopping effort through wealth effects. The force of this latter channel depends on the inverse intertemporal elasticity of substitution σ and the habit stock parameter h.

3.2. Capital goods suppliers' problem

The capital goods supplier searches for investment goods, transforms them into capital, and then costlessly differentiates them into specific capital goods for each sector. The supplier chooses how much to invest, the amount of workers involved in shopping, the allocations of capital, and the utilization rates.

$$\max_{l_{kt},i_t,u_t^C,u_t^I,k_{t+1}} \sum_{t=0}^{\infty} \beta^t \lambda_t [r_t^C u_t^C k_t^C + r_t^I u_t^I k_t^I - w_t l_t - P_t^I i_t] \quad \text{s.t.}$$

$$P_t^I i_t = \int_0^{l_t \zeta A (Q_t^I)^{1-\phi}} p_{jt}^i i_{jt} dt \tag{13}$$

$$k_{t+1} \leq (1-\delta^C(u_t^C))k_t^C + (1-\delta^I(u_t^I))k_t^I + i_t - \Phi\left(\frac{i_t}{k_t}\right) k_t \tag{14}$$

$$k_t = k_t^C + k_t^I \tag{15}$$

Equation (13) links the choice of individual investment units and shopping effort to the aggregate investment good. Equation (14) is the law of motion for capital accounting for sector specific depreciation rates and quadratic adjustment costs

$$\Phi(x) = \frac{\psi_K}{2}(x - \delta^k)^2$$

Finally, (15) says that that each type of capital must add up to the total capital stock.

Let λ_t be the Lagrange multiplier on (13) and μ_t the multiplier on (14). (Appendix B.2) solves the problem in detail where we obtain the following optimality conditions:

$$q_t = \frac{P_t^I}{1 - \Phi'\left(\frac{i_t}{k_t}\right)} \tag{16}$$

$$q_t \delta^{i'}(u_t^i) = r_t^i, \quad i \in \{C, I\} \tag{17}$$

$$q_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^i u_{t+1}^i + q_{t+1} \left(1 - \delta^i(u_t^i) + \Phi^{i'}\left(\frac{i_{t+1}}{k_{t+1}}\right) \frac{i_{t+1}}{k_{t+1}} - \Phi^i\left(\frac{i_{t+1}}{k_{t+1}}\right) \right) \right] \right\} \tag{18}$$

$$w_t l_t = \frac{P_t^I i_t}{\varepsilon_i - 1} \tag{19}$$

where $q_t \equiv \eta_t/\lambda_t$ is the relative price of capital. (16) is the standard pricing equation for capital, where adjustment costs create a wedge between the cost of obtaining capital P_t^I and the value it brings to the firm out of steady state. (17) says that capital good suppliers increase utilization up to the point that the additional rental income just offsets the higher depreciation. (18) is the Euler equation with respect to capital. Since capital good suppliers can costlessly differentiate capital either toward the investment or consumption sectors, the Euler equation can be expressed equivalently in terms of either one. Optimal capital accumulation guarantees that the rate of return on capital net of depreciation is the same in both sectors. (19) gives the optimal employment of workers to shop for investment goods.

3.3. Firms

Firms choose whether to operate in the consumption or investment sector, which involve separate submarkets. Given demand for y_i units of goods at a location, the firm in submarket i needs to dispatch variable labor equal to

$$h^i(y_i, k, l_1) = \left(\frac{y_i}{Z^i (uk)^\alpha l_1^{\alpha_1}} \right)^{1/\alpha_2}$$

The demand schedule can be described as $y_i = (\rho_j)^{-\varepsilon} y$ due to CES preferences over varieties. The variable labor that the firm needs upon installing k and l_1 in each location is $l_2 = A(Q^i)^{-\phi} h^i(y_i, k, l_1)$. This fact implies that the necessary inputs to satisfy demand are

$$Z^i (uk)^\alpha l_1^{\alpha_1} l_2^{\alpha_2} = (A(Q^i)^{-\phi})^{\alpha_2} y_i$$

The problem of a firm which specializes in selling to consumers is

$$\max_{c_{it}, p_{it}, l_{1t}, l_{2t}, l_{kt}, i_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \lambda_t [A(Q_t^H)^{-\phi} c_{it} p_{it} - w_t (l_{1t} + l_{2t}) - r_t^C u_t^C k_t^C] \quad \text{s.t.}$$

$$c_{it} = \left(\frac{p_{it}}{P_t^C} \right)^{-\varepsilon} c_t \tag{20}$$

$$(A(Q_t^H)^{-\phi})^{\alpha_2} c_{it} \leq Z_t^C (u_t k_t)^\alpha l_{1t}^{\alpha_1} l_{2t}^{\alpha_2} \tag{21}$$

Equation (20) is the demand curve and equation (21) is the input requirement to satisfy demand, which depends on technology, goods market frictions, and the utilization of capital. The problem is solved by first obtaining the cost-minimizing input choice and then obtaining the profit-maximizing price given the cost-minimizing bundle. Given output $\overline{y}$, the cost minimization problem of the retailer is

$$\min w(l_1 + l_2) + r^C u^C k \quad \text{s.t.}$$

$$A(Q^h)^{-\phi} c_i \geq \overline{y}$$

$$Z^C (u^C k)^\alpha l_1^{\alpha_1} l_2^{\alpha_2} = (A(Q^h)^{-\phi})^{\alpha_2} c_i$$

Substitute the production constraint into the input requirement and let $y = \overline{y}^{\alpha_2}$ to rewrite the problem as

$$\min w(l_1 + l_2) + r^C u^C k \quad \text{s.t.}$$

$$Z^C c_i^{\alpha_2 - 1} (u^C k)^\alpha l_1^{\alpha_1} l_2^{\alpha_2} \geq y$$

First order conditions are

$$[l_1] : w = \frac{\alpha_1 y}{l_1} MC \tag{22}$$
$$[l_2] : w = \frac{\alpha_2 y}{l_2} MC$$
$$[k] : r^C = \frac{\alpha y}{u^C k} MC \tag{23}$$

The ratio of the two labor demand conditions implies $l_1 = \alpha_1/(1-\alpha)l$ and $l_2 = \alpha_2/(1-\alpha)l$. The standard pricing condition for a monopolistic competitor yields

$$p = \frac{\varepsilon}{\varepsilon - 1} MC \tag{24}$$

Multiplying both numerator and denominator of (22) – (B.10) by N^H and using the pricing rule (24) yields

$$w = \frac{(1-\alpha)C}{\frac{\varepsilon}{\varepsilon-1} L^H}, \quad r^C = \frac{\alpha C}{\frac{\varepsilon}{\varepsilon-1} u^C K^H} \tag{25}$$

where L^H and K^H is the total retail labor and capital used in selling to households. Since a condition analogous to (25) holds for retailers selling to the government, summing yields

$$w = \frac{(1-\alpha)Y^C}{\frac{\varepsilon}{\varepsilon-1} L^C}, \quad r^C = \frac{\alpha Y^C}{\frac{\varepsilon}{\varepsilon-1} u^C K^C}$$

The relationship between retail output for consumers in terms of inputs satisfies

$$\begin{aligned} C &= N^H p A (Q^H)^{-\phi} c_i \\ &= N^H p A (Q^H)^{-\phi} \frac{Z^C (u^C k^H)^{\alpha} (l_1^H)^{\alpha_1} (l_2^H)^{\alpha_2}}{(A(Q^H)^{-\phi})^{\alpha_2}} \\ &= (A(Q^H)^{-\phi})^{1-\alpha_2} p Z^C (u^C K^H)^{\alpha} (L^H)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \end{aligned} \tag{26}$$

Similarly,

$$G = (A(Q^G)^{-\phi})^{1-\alpha_2} p Z^C (u^C K^G)^{\alpha} (L^G)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \tag{27}$$

Using $L^H/L^G = K^H/K^G = C/G$, we find that tightness is equalized across submarkets.

Lemma 1.

$$Q^H = Q^G$$

Thus, $C/G = S^H/S^G$. For intuition, consider that firms face the same labor and capital costs regardless of whether they sell to consumers or the government, enjoy the same price-setting power (elasticity of substitution), and are free to switch among them. In order for both markets to be active, they must have the same tightness. The profit of a firm selling to consumers satisfies py^h/ε. Multiplying by N^H and rearranging yields the profit equation for the household sector:

$$d = \frac{C}{\varepsilon N^H}$$

Hereafter, we use Q^C to refer to the common market tightness. Similarly, the profit of a firm selling to the government sector is $G/(\varepsilon N^g)$. The free mobility condition says that firms switch sectors until profits are equalized:

$$d = \frac{C}{\varepsilon N^H} = \frac{G}{\varepsilon N^G} \Leftrightarrow \frac{N^G}{N^H} = \frac{G}{C}$$

In the steady state, $N^G/N^H = g_c$.

The problem of the investment firm can be examined symmetrically. Skipping steps, we find that the input choices satisfy

$$w = \frac{(1-\alpha)I}{\frac{\varepsilon_i}{\varepsilon_i - 1}L^I}, \quad r^I = \frac{\alpha I}{\frac{\varepsilon_i}{\varepsilon_i - 1}u^I K^I}$$

We can write investment output as

$$\begin{aligned} I &= N^I q A(Q^I)^{-\phi} i_j \\ &= A(Q^I)^{-\phi} \frac{Z^I (u^I k^I)^\alpha (l_1^I)^{\alpha_1} (l_2^I)^{\alpha_2}}{(A(Q^I)^{-\phi})^{\alpha_2}} \\ &= (A(Q^I)^{-\phi})^{1-\alpha_2} q Z^I (u^I K^I)^\alpha (L^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \end{aligned}$$

3.4. Government

The government faces a static problem of choosing consumption of each variety and shopping effort so as to maximize utility given allowable expenses G:

$$\max_{S^g, g_i} g(g_i, S^g)^{1-\eta} - \kappa S^g \quad \text{s.t.}$$

$$P(s^g) g(g_i, S^g) = G$$

The first order conditions are

$$[g_i] : \frac{\partial g}{\partial g_i}[1 - \lambda^g P^g] = 0 \tag{28}$$

$$[s^g] : \frac{\partial g}{\partial S^g} - \kappa - \lambda^g \frac{\partial P g}{\partial S^g} = 0 \tag{29}$$

The first order condition (28) implies $g^{-\eta} = \lambda^g P^g$. Using the quantity aggregator for the government and the price index, we find

$$\frac{\partial g}{\partial S^g} = \left(\frac{\varepsilon}{\varepsilon - 1}\right) \rho_i^g A Q^{1-\phi} g_i$$
$$\frac{\partial P g}{\partial S^g} = A Q^{1-\phi} p_i g_i$$

Plugging these into (29), we find

$$\begin{aligned} \kappa &= \left[\frac{\varepsilon}{\varepsilon - 1}\rho^g A (Q^g)^{1-\phi} g_i - A(Q^g)^{1-\phi}\rho^g g_i\right] \\ &= \left[\frac{\rho^g A (Q^g)^{1-\phi} g_i}{\varepsilon - 1}\right] \end{aligned}$$

Multiplying both sides by S^g, we can rearrange this in terms of shopping effort and the government consumption bundle:

$$S^g = \frac{g^{1-\eta}}{(\varepsilon - 1)\kappa} \tag{30}$$

The only role of (30) is to determine the composition of g and P^g given government purchases: $G = P^g g$. Given that government purchases follow an exogenous process, and that P^g and g are not otherwise linked to variables of interest, (30) is not relevant for the estimation. Thus, it would have been equivalent to consider more general preferences, such as constant relative risk aversion.

3.5. Aggregation

In the aggregate, the total number of shares $x_t = 1$, so that ν_t represents the stock market capitalization after paying out dividends. Given a unit measure of firms, d_t also represents aggregate profits.

We can aggregate consumption and investment in physical capital as $C_t = P_t^H c_t$ and $I_t = P_t^I i_t$. As capital goods suppliers are perfectly competitive, revenues equal input costs:

$$\begin{aligned} r_t^C u_t^C K_t^C + r_t^I u_t^I K_t^I &= w_t L_t^K + I_t \\ &= \frac{\varepsilon_i}{\varepsilon_i - 1} I_t \end{aligned}$$

The spread between investment and rental income exactly covers the labor search costs required in shopping for investment goods.

Then we can write aggregate retail consumption by summing (26) and (27) and using Lemma 1:

$$Y^C = (A(Q^H)^{-\phi})^{1-\alpha_2} p Z^C (u^C K^C)^{\alpha} (L^C)^{1-\alpha} \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$

$$I = (A(Q^I)^{-\phi})^{1-\alpha_2} q Z^I (u^I K^I)^{\alpha} (L^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$

Sector-specific total factor productivity is $\mathcal{Z}^i = Y^i/(K^i)^{\alpha}(L^i)^{1-\alpha})$, or

$$\mathcal{Z}^C = pZ^C (A(Q^C)^{-\phi})^{1-\alpha_2} (u^C)^{\alpha} \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$
$$\mathcal{Z}^I = qZ^I (A(Q^I)^{-\phi})^{1-\alpha_2} (u^I)^{\alpha} \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$

The profit share in each sector is the inverse of the elasticity of substitution, so that aggregate profits satisfy

$$d = Y^C/\varepsilon + I/\varepsilon_i.$$

We aggregate (6) across households:

$$Y_t = C_t + I_t + \nu_t N_{E,t} + T_t = w_t L_t + N_t d_t + r_t^K (u_t^C K_t^C + u_t^I K_t^I + K_t^E)$$

Using the government budget constraint, $G_t = T_t$, we obtain

$$Y_t = C_t + I_t + \nu_t N_{Et} + G_t$$

3.6. Steady state

Appendix C.3 derives the main steady-state relationships in the model. Here, we discuss a few major properties. A steady state equilibrium requires that investment just offsets depreciation: $I = \delta^K K$ and the Euler equation for capital implies $r_E^K = r + \delta^K$.

The shares of profits in the consumption and investment sectors is

$$\frac{N^C d}{Y^C} = \frac{1}{\varepsilon}$$
$$\frac{N^I d}{I} = \frac{1}{\varepsilon_i}$$

The composition of firms in each sector is

$$N^C = \frac{r^K \varepsilon_i - \delta^K \alpha(\varepsilon_i - 1)}{r^K \varepsilon_i + \delta^K \alpha(\varepsilon - \varepsilon_i)}$$
$$N^I = \frac{\alpha \delta^K (\varepsilon - 1)}{r^K \varepsilon_i + \delta^K \alpha(\varepsilon - \varepsilon_i)}$$

The share of capital in each sector to the overall capital stock is

$$\phi^I \equiv \frac{K^I}{K} = \frac{\varepsilon_i - 1}{\varepsilon_i} \frac{\alpha \delta^K}{r + \delta^K}$$
$$\phi^C \equiv \frac{K^C}{K} = \frac{\varepsilon_i r^K - \alpha \delta^K (\varepsilon_i - 1)}{\varepsilon_i (r + \delta^K)}$$

The labor share in the investment shopping sector is

$$\phi^{LK} \equiv \frac{L^K}{L} = \frac{\alpha \delta^K}{\alpha \delta^K + (1-\alpha)(r + \delta^K)(\varepsilon_i - 1)}$$

Thus, the labor shares in consumption and investment satisfy

$$\frac{L^C}{L} = \phi^C (1 - \phi^{LK}), \quad \frac{L^I}{L} = \phi^I (1 - \phi^{LK}),$$

Aggregate profits equal d given the fact that there is a unit mass of firms. The profit share of output is

$$\frac{d}{Y} = \frac{r^K \varepsilon_i + \alpha \delta^K (\varepsilon - \varepsilon_i)}{r^K \varepsilon \varepsilon_i + \alpha \delta^K (\varepsilon - \varepsilon_i)}$$

If we let $\delta^K \to 0$, then output coincides with retail sales, and the profit share approaches $1/\varepsilon$. Moreover, if $\varepsilon = \varepsilon_i$, then the consumption and investment sectors have an overall profit share of $1/\varepsilon$.

3.7. The unadjusted and adjusted Solow residuals

The Solow residual is the ratio of aggregate output to share-weighted inputs $K^{1-\omega} L^{\omega}$, where ω is the labor share of income. Even though the elasticity of output with respect to labor is $1 - \alpha$ in each sector, some labor also accrues to workers searching for differentiated investment goods.

We now proceed to derive the Solow residual writing output, using the expenditure approach, and substituting the corresponding production functions for each sector:

$$\begin{aligned}
Y &= Y^C + I \\
&= (A(Q^H)^{-\phi})^{1-\alpha_2} p Z^C (u^C K^C)^{\alpha} (L^C)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \\
&\quad + (A(Q^I)^{-\phi})^{1-\alpha_2} q Z^I (u^I K^I)^{\alpha} (L^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}
\end{aligned}$$

We define the Solow residual and the Solow residual adjusted for utilization along both the extensive margin (shopping effort) and intensive margin (capital utilization rate). To control for changes in relative prices, we use base-year prices p^* and $q*$. Specifically, we define

$$\begin{aligned} Y^{obs} &= (A(Q^H)^{-\phi})^{1-\alpha_2} p^* Z^C (u^C K^C)^\alpha (L^C)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \\ &\quad + (A(Q^I)^{-\phi})^{1-\alpha_2} q^* Z^I (u^I K^I)^\alpha (L^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \\ Y^{util,obs} &= (A(Q^{*H})^{-\phi})^{1-\alpha_2} p^* Z^C (K^C)^\alpha (L^C)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \\ &\quad + (A(Q^{*I})^{-\phi})^{1-\alpha_2} q^* Z^I (K^I)^\alpha (L^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}} \end{aligned}$$

since $u^{*C} = u^{*I} = 1$. Here Q^* and u^* denote fixing goods market tightness and capital utilization at their steady state values respectively. In this way, we control for movements in variables unrelated to technology to produce a purified Solow residual and measure the relative contribution of goods market frictions and capital utilization to observed movements in TFP. We define the Solow residual and utilization-adjusted Solow residuals as follows:

$$\mathcal{Z} = \frac{Y^{obs}}{K^{1-\omega} L^\omega}, \quad \mathcal{Z}^{util} = \frac{Y^{util,obs}}{K^{1-\omega} L^\omega}$$

where ω is the steady-state labor share of income. The steady-state labor share of income satisfies

$$\omega \equiv \frac{wL}{Y} = \frac{r^K(\varepsilon - 1)}{\frac{\alpha\delta^K(\varepsilon-\varepsilon_i)}{\varepsilon_i} + r^K\varepsilon} \frac{\frac{\alpha\delta^K}{r^K(\varepsilon_i-1)} + 1 - \alpha}{\frac{\alpha\delta^K}{r^K(\varepsilon_i-1)} + 1} \tag{31}$$

Several special cases are worth considering. First, consider $\varepsilon_i \to \infty$. Then the labor share approaches

$$(1-\alpha)\frac{r^K(\varepsilon-1)}{r^K\varepsilon - \alpha\delta^K}$$

No labor is employed in shopping for investment goods, so the share $1-\alpha$ is deflated by the average markup between the two sectors. In particular, as $\delta^K \to 0$, the labor share approaches $(1-\alpha)(\varepsilon-1)/\varepsilon$. If both $\varepsilon, \varepsilon_i \to \infty$, then of course the labor share approaches $1-\alpha$. Finally, if $\varepsilon = \varepsilon_i$ at a finite value, then the labor share is both deflated by the average markup but also inflated by the role of workers shopping for investment goods.

4. Quantitative analysis

4.1. Bayesian estimation

The use of Bayesian estimation is natural for three reasons. First, with important exceptions, there are few direct ways of identifying the shocks.[3] Estimating the relative contributions of the shocks is an important objective and is implementable via the forecast error variance decomposition. Second, the are several parameters which are very important for the transmission mechanism but uninformed by prior studies or steady-state targets, especially the matching function elasticity ϕ. Third, we can quantify parameter uncertainty by incorporating probability bands in the impulse responses.

I discuss the procedure very briefly as An and Schorfheide (2007) and Herbst and Schorfheide (2015) provide detailed expositions. First, I set a joint prior distribution $P(\Theta)$. Level parameters do not affect the first-order dynamics, and thus are excluded from Θ. I also fix several parameters. I set $\beta = 0.99$, consistent with an annual real interest rate of 4%, $\delta_K = 0.025$, which is consistent with 10% annual depreciation of physical capital. Moreover, I set $\delta = 0.025$, which approximates an average product destruction rate of 9% from Bernard, Redding, and Schott (2010).

The steady-state labor share (31) plays a key role in the estimation strategy. Following Bai, Rios-Rull, and Storesletten (2012), we ensure that the steady-state labor share ω matches the target in the data. Specifically, given calibrated values of δ, δ^K and r together with draws of ε and ε_i, we choose α so that it matches a labor share of income of 62%.[4] Since the substitution elasticities ε and ε_i are random variables from a Bayesian perspective, so is α.

[3] Major exceptions include the approach of Basu, Fernald, and Kimball (2006) for technology shocks, that of Greenwood, Hercowitz, and Huffman (1988) for investment-specific productivity shocks, and substantial work in identifying monetary policy, government spending, and news shocks. A few key references are Romer and Romer (2004) and Swanson (2015) for monetary policy shocks, Blanchard and Perotti (2002) and Ramey (2011) for government spending shocks, and Barsky and Sims (2011) for news shocks, but there are many more.

[4] I measure labor share of income using the FRED code LABSHPUSA156NRUG. The average between 1948 and 2009 is 62%.

Parameter	Value	Interpretation
β	0.99	Discount factor
δ	0.025	Firm exit rate
δ_K	0.025	Capital depreciation rate
α	Value consistent with $\omega = 0.62$	Elasticity of sectoral output with respect to capital

Table 2: Calibrated parameters. Here α is implicitly a random variable, since ε is a random variable and α varies with ε so as to match a labor income share of 62%.

Before proceeding, we verify that the model implies reasonable ratios of major quantities with respect to output. We examine the labor, rental, and profit shares of income; the consumption and investment shares; the composition of firms; and the capital share of output and labor in investment goods shopping.

wL/Y	$r^K K/Y$	d/Y	N_I	N_C	C/Y	G/Y	I/Y	$K/(4Y)$	ϕ_{LK}
0.62	0.16	0.22	0.066	0.934	0.689	0.194	0.116	1.164	0.026

Table 3: Ratios of expenditure sources, income sources, and the capital stock relative to output. For the latter, output has been annualized by multiplying by 4. Here we set $\varepsilon = 4.3$ and $\varepsilon_i = 8.0$.

In general, the ratios are empirically realistic, with a consumption share of output near 70%, a profit share of 22%, and a capital-stock-to-GDP in excess of 1. Again, the only targeted quantity here is the labor share of income. Less than 3% of labor is used for investment shopping

To prevent stochastic singularity and obtain a well-defined likelihood function, we need to ensure that no observable series can be expressed as a function of others. A necessary condition is that there are as many shocks as observables series, as explained by Ruge-Murcia (2007).[5] As Offick and Winkler (2019), we add measurement errors to wages and investment. Additionally, we include measurement error in labor hours.

[5]To grasp the problem of stochastic singularity, consider a simple real business cycle model with an unobserved technology series and consumption and output used as observables. In the reduced form VAR(1) solution, the shocks in each equation are just multiples of each other. This finding, in turn, implies that certain ratios of observed variables are constant. Thus, one observable can be inferred deterministically from

The shock structure is as follows:

- General and independent sector-specific technology shocks

$$u_{ZC} = \epsilon_Z + \epsilon_{ZC}$$
$$u_{ZI} = \epsilon_Z + \epsilon_{ZI}$$
$$Z^C = \rho_{ZC} Z_{C,-1} + u_{ZC}$$
$$Z^I = \rho_{ZI} Z_{I,-1} + u_{ZI}$$

- Correlated shopping demand shocks

$$u_\zeta = \epsilon_{shop} + \epsilon_\zeta$$
$$u_\kappa = \gamma_{shop}\epsilon_{shop} + \epsilon_\kappa$$
$$\zeta = \rho_\zeta \zeta_{-1} + u_\eta$$
$$\kappa = \rho_\kappa \kappa_{-1} - u_\kappa$$

- Labor supply follows an AR(2) process

$$\chi = \rho_{1,\chi}\chi_{-1} + \rho_{2,\chi}\chi_{-2} + \epsilon_\chi$$

- Other shocks are independent AR(1) processes

$$x = \rho_x x_{-1} + \varepsilon_x \quad \text{for} \quad x \in \{\theta, b, G\}$$

The baseline model has 10 shocks $(\epsilon_Z, \epsilon_{ZC}, \epsilon_{ZI}, \epsilon_{shop}, \epsilon_\zeta, \epsilon_\kappa, \epsilon_\theta, e_\chi, \epsilon_b, \epsilon_G)$ for 9 observable variables $(C, TI, Y, L, w, G, \mathcal{Z}, \mathcal{Z}^{util}, q_p)$. Table 4 summarizes main specification of the estimation.

the other. However, this relationship does not hold in the data, so fitting the data is impossible without another stochastic disturbance.

Estimation feature	
Time period	1960Q1-2022Q4
Filtering	Hamilton regression filter ($h = 8, p = 4$)
Observables	$(C, TI, Y, L, w, G, \mathcal{Z}, \mathcal{Z}^{util}, q_p)$
Shocks	$(\epsilon_Z, \epsilon_{ZC}, \epsilon_{ZI}, \epsilon_{shop}, \epsilon_\zeta, \epsilon_\kappa, \epsilon_\theta, e_\chi, \epsilon_b, \epsilon_G)$
Measurement error	$(\sigma_{w,ME}, \sigma_{TI,ME}, \sigma_{L,ME})$

Table 4: Specifications for estimation

The composite Θ of parameters to estimate excludes those which do not affect the first-order dynamics and those calibrated directly. It is conceptually useful to decompose Θ into three blocks:

$$\Theta_1 = (\psi, \gamma, \sigma, \phi, \varepsilon, \varepsilon_i, h, \alpha_2, \Phi_C, \Phi_I, \gamma_{shop})$$

$$\Theta_2 = (\rho_{ZC}, \rho_{ZI}, \rho_\zeta, \rho_\kappa, \rho_{1,\chi}, \rho_{2,\chi}, \rho_\theta, \rho_A, \rho_b, \sigma_Z, \sigma_{ZC}, \sigma_{ZI}, \sigma_{shop}, \sigma_\zeta, \sigma_\kappa, \sigma_\theta, \sigma_\chi, \sigma_b, \sigma_G)$$

$$\Theta_3 = (\sigma_{w,ME}, \sigma_{TI,ME}, \sigma_{L,ME})$$

Start updating from here

and $\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3$. Here, Θ_1 are the standard model parameters which affect first-order dynamics, Θ_2 are the autoregressive and conditional-standard deviation parameters of the shocks, and Θ_3 are the measurement errors.

The next step is to recast the model in linear state space form. Accordingly, Table C.7 summarizes the log linear system. For a linearized model, the likelihood function can be computed using the Kalman filter, which generates optimal predictions and updates of the unobservable variables given the data. We first maximize the posterior density, and then use the Metropolis Hastings algorithm to sample the posterior distribution. We simulate 500, 000 draws with a burn-in of 20%, which suffices given the rapid convergence to the posterior distribution.

4.2. Taking the model to the data

In contract to a one-sector model, a multisector model of the type studied here raises the problem of aggregating sectoral quantities into economy-wide quantities, such as output and the Solow residual. Though we use the investment basket as the numeraire, quantity movements are not generally independent of the choice of numeraire. For instance, if there is a positive shock to Z^C, the relative price of consumption goods falls. In terms of the investment good, consumption may fall even though the actual units purchased rises. However, if the consumption good were the numeraire, this effect does not exist and instead investment goods rise in price. Thus, a Z^C shock is contractionary with the investment good as the numeraire and expansionary with the consumption good as the numeraire. The reasoning is symmetric with a positive Z^I shock.

Of course, quantity movements ought to be invariant to the choice of numeraire. We overcome this issue by using base-year prices as in Bai, Rios-Rull, and Storesletten (2012) and Huo and Ríos-Rull (2018). [6] We also need to make a second adjustment, some changes in the consumption and investment baskets C_t and I_t occur because of product variety. by Bilbiie, Ghironi, and Melitz (2012) argue for removing variety effects since CPI data does not correspond to a welfare-based price index like P_t.

To be concrete, we measure output using base-year prices p_0 and q_0:

$$Y = p_0 Y^C + q_0 I$$

Moreover, since we log linearize around a steady state, we set the base-year prices at steady-state values: $p = p^*$ and $q = q^*$. Finally, we include measurement errors. The log-linearized

[6] Duernecker, Herrendorf, Valentinyi, et al. (2017) discuss the measurement problem carefully in the context of a multisector growth model and demonstrate that a chain-weighted (Fisher) index generates quantity movements independent of both the numeraire and the use of current-period or base-period prices. However,the indices are first-order equivalent.

measurement equations we get are as follows:

$$
\begin{aligned}
C^{obs} &= C - p \\
G^{obs} &= G - p \\
I^{obs} &= I - q + \varepsilon_{I,ME} \\
Y^{obs} &= Y_Y^C(Y^C - p) + I_Y(I - q) \\
Y^{util,obs} &= Y_Y^C(Y^{C,util} - p) + I_Y(I^{util} - q) \\
SR^{obs} &= Y^{obs} - (1-\omega)K - \omega L \\
SR^{util,obs} &= Y^{util,obs} - (1-\omega)K - \omega L \\
q_p &= q - p \\
L^{obs} &= L + \epsilon_{L,ME} \\
w^{obs} &= w + \epsilon_{w,ME}
\end{aligned}
$$

4.3. Posterior distribution and identification

Table 5 show prior and posterior distributions, obtained by the Metropolis-Hastings algorithm, for all structural parameters and shocks. The estimated parameters are well identified with meaningfully different prior and posterior means, and low standard errors with the exception of the share of variable labor in the consumption sector ($varshare$) and variance of the investment-sector shopping technology (ϵ_ζ). Notably, the intratemporal preference process (θ) and sector-specific productivity processes (Z^C, Z^I) are estimated to be highly persistent with AR(1) coefficients equal to $0.954, 0.909$ and 0.895, respectively. Other shock processes, including household shopping disutility, investment shopping efficiency, intertemporal utility, labor supply, and government spending are moderatly persistent. Taken together, the estimated shock processes suggests that most shocks can evenly compete in explaining forecast errors at longer time horizons, while we expect the relative importance of technology shocks and intratemporal preference shocks to grow over time. The estimated variance of the common technology shocks are slightly higher compared to the sector-specific components, suggesting specifying a common component to technology processes is important in explaining the data.

Turning to the structural parameters, the share of variable labor in the consumption sector is estimated to be 34.9 percent, which gives a substantial role to pre-installed factors in affecting the extensive margin of utilization. The posterior means for elasticity of substitution across varieties are 12.9 and 13.6 for the consumption and investment sectors, respectively, suggesting that variable shopping intensity in both sectors is relevant. The intertemporal elasticity of substitution is estimated to be 1.71 which is within a reasonable range of estimates obtained in the literature. The elasticity of rental rate with respect to utilization have posterior means of 0.94 and 0.50 for the consumption sector and investment sector respectively, indicating that the intensive margin of utilization adjustments play an important role. Notably, the estimated value of γ, which measures role of wealth effects on labor supply, is very small at 0.075. Wealth effects play very little role on labor supply, and the estimated preferences are approximately those obtained under a Greenwood–Hercowitz–Huffman specification. This finding is consistent with the literature on news shocks in business cycle models such as Schmitt-Grohé and Uribe (2012) . The elasticity of the matching function with respect to aggregate search intensity is estimated to be 0.25.

Parameter	Distribution	Prior Mean	Posterior Mean	Prior Std	Posterior Std
ψ	Gamma	2.389	3.195	0.4	0.329
ϵ	Gamma	3.8	12.85	1.0	1.175
ε_i	Gamma	3.8	13.596	1.0	1.311
σ	Gamma	1.5	1.712	0.25	0.321
ψ_C	Beta	0.5	0.942	0.15	0.032
ψ_I	Beta	0.5	0.501	0.15	0.11
$varshare$	Beta	0.5	0.349	0.2	0.162
γ	Beta	0.25	0.075	0.15	0.075
ϕ	Beta	0.5	0.249	0.25	0.065
ψ_K	Gamma	1.57	7.359	1.5	2.324
λ_1	Beta	0.6	0.888	0.2	0.027
λ_2	Beta	0.0	-0.027	0.2	0.066
ρ_θ	Beta	0.6	0.954	0.2	0.027
ρ_{Z^C}	Beta	0.6	0.895	0.2	0.024
ρ_{Z^I}	Beta	0.6	0.909	0.2	0.02
ρ_b	Beta	0.6	0.821	0.2	0.139
ρ_κ	Beta	0.6	0.831	0.2	0.028
ρ_ζ	Beta	0.6	0.832	0.2	0.031
ρ_g	Beta	0.6	0.885	0.2	0.023
e_χ	Inverse Gamma	0.01	0.04	0.004	0.005
e_κ	Inverse Gamma	0.01	0.009	0.004	0.002
e_θ	Inverse Gamma	0.01	0.039	0.004	0.01
e_{shop}	Inverse Gamma	0.01	0.053	0.004	0.013
e_ζ	Inverse Gamma	0.01	0.008	0.004	0.002
e_b	Inverse Gamma	0.01	0.007	0.004	0.002
e_Z	Inverse Gamma	0.01	0.01	0.004	0.0
e_{ZC}	Inverse Gamma	0.01	0.004	0.004	0.0
e_{ZI}	Inverse Gamma	0.01	0.006	0.004	0.001
e_g	Inverse Gamma	0.01	0.016	0.004	0.001
e_{LME}	Inverse Gamma	0.01	0.026	0.004	0.001
e_{TIME}	Inverse Gamma	0.01	0.05	0.004	0.002
e_{wME}	Inverse Gamma	0.01	0.033	0.004	0.002

Table 5: Prior and posteriors for model parameters.

4.4. Forecast error variance decomposition

We next examine the portion of the forecast error variance that can be attributed to each structural shock at a particular horizon. Figure 3 decomposes the forecast error variance among the 10 shocks 1 quarter, 1 year, 2 year, and 10 year time horizons. Whereas the first three horizons correspond to short-run effects, the last can be regarded as a medium-run effect. Variation in output is driven predominantly by the common component of shopping efficiency and production technology as well as labor supply. Intratemporal prefrence shocks play a smaller role, and sector-specific technology shocks are negligible. Given that the production technology processes were estimated to be highly persistent, their relative importance increases at longer time horizons, disfavoring the role of shopping efficiency. A similar pattern holds for investment, except that the relative importance of shopping efficiency actually increases from 1 quarter to 1 year due to the sluggish response of investment from adjustment costs.

Interestingly, shopping efficiency explains nearly half of the variance in the Solow residual at 1 quarter, and continues to explain over one-third of the variance at all time horizons. Shopping efficiency affects the total demand that firms observe, and thereby influences capacity utitilization which in turn affects the Solow residual. This is the way in which demand can influence the naive measure of technology—the Solow residual. If, however, we use the utilization-adjusted Solow residual time series, then we see that this demand channel dissapears and the variance is entirely explained by common production technology and the consumption-sector production technology. This interpretation is particularly clear in Figure 4 where the shocks are bundled into four categories: shopping demand, standard demand, technology, and labor supply. Here we see that the utilization adjusted Solow residual is entirely explained by technology shocks, while the unadjusted Solow residual is equally explained by technology and demand shocks and all time horizons.

4.5. Historical decomposition

Figure 5 shows the historical contribution contribution of each of the 10 shock processes in explaining deviations of output and the Solow residual from their steady-state values. To simplify interpretation, 6 bundles the shocks into the four categories defined earlier. This allows us to examine the contribution of shocks in concrete episodes such as the Volcker recession, dotcom bust, Great Recession, and pandemic recession.

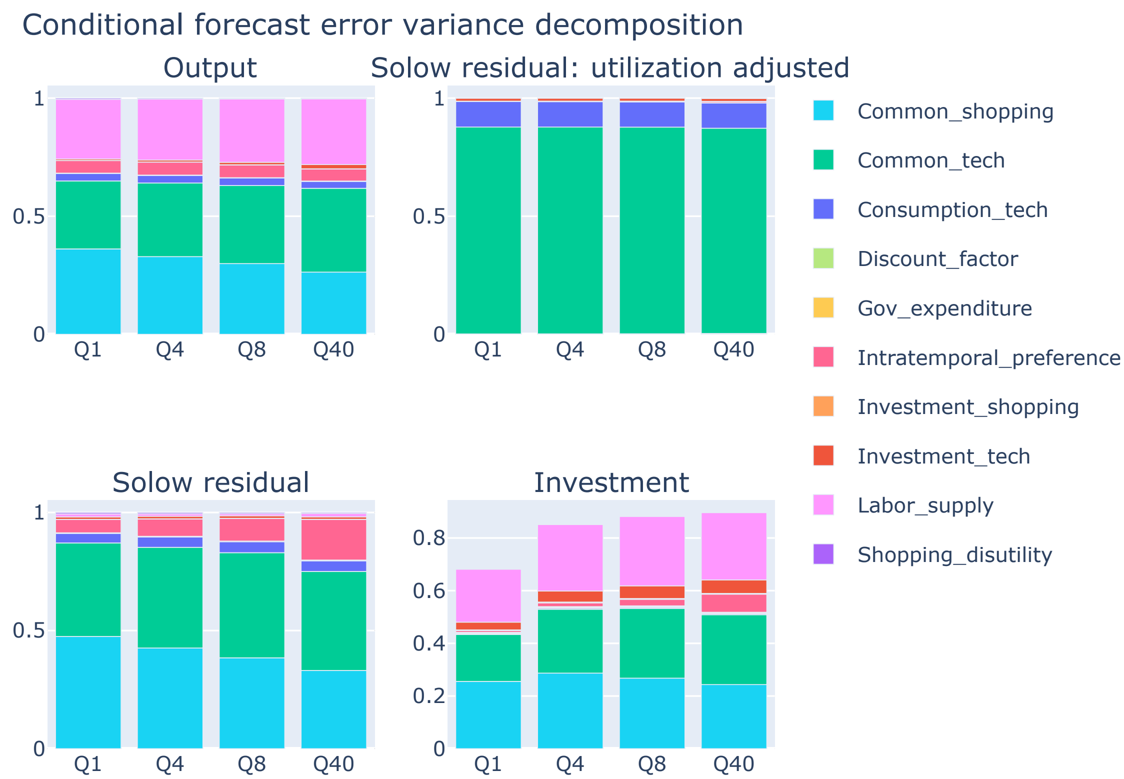

Figure 3: Forecast error variance decomposition across all 10 shock processes. Bar plots may not sum to 1 because of measurement errors, which are not displayed.

Firstly, shopping demand plays a more prominent role in historical deviations compared to standard demand channels. Moreover, shopping demand was important in every recession over the sample period. The same cannot be said about technology shocks. In particular, technology shocks actually play a positive role in output deviations throughout late 1990s and early 2000s, in spite of the 2001 recession. Rather, shopping technology and labor supply shocks are required to explain the recession in 2001. A similar pattern holds for the most recent recession in 2020—technology shocks are in fact positive whereas a combinantion of demand shocks and labor supply shocks account for the sharply negative output gap.

The Solow residual is primarily driven by technology and shopping efficiency shocks with small roles attributed to labor supply and intratemporal preference shocks. During some recessions, for example the 2007 financial crisis, both technology and shopping efficiency shocks are needed to explain the negative output gap. In other periods however, such as the 2001 recession and the 2020 recession, technology shocks are positive so shopping demand shocks are essential in explaining the business cycle troughs. We observe that shopping demand

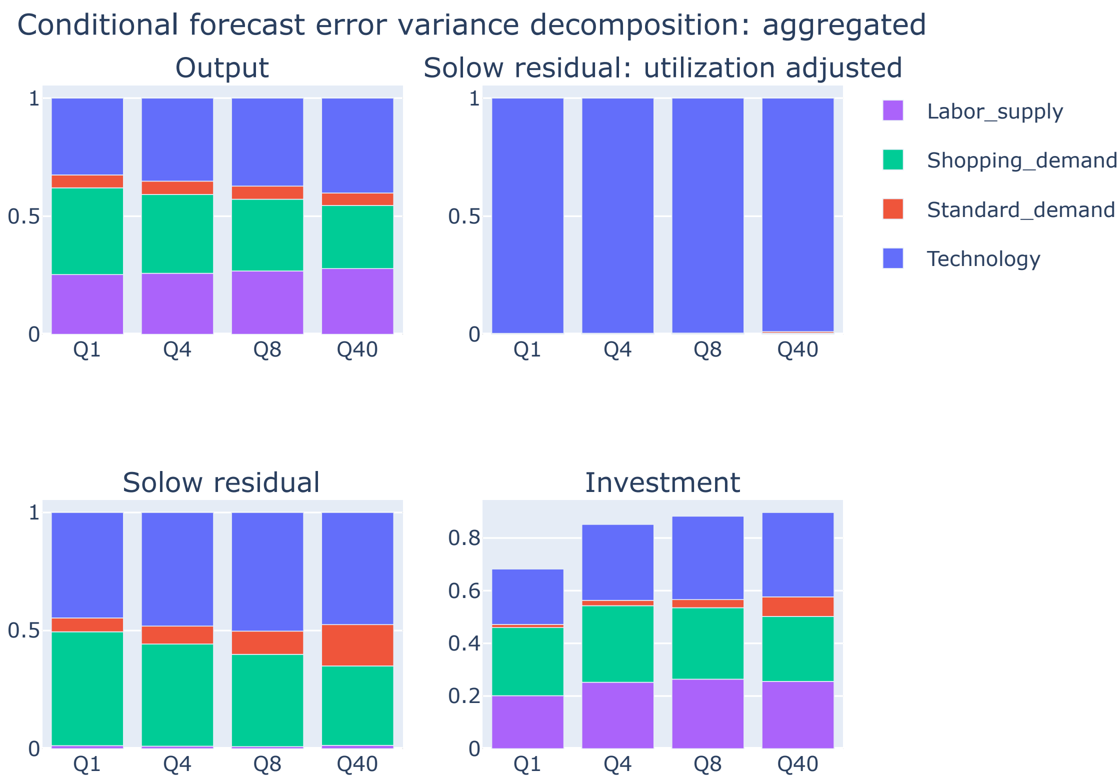

Figure 4: Forecast error variance decomposition across 4 categories of shocks: Shopping Demand = Common Shopping + Investment Shopping + Shopping Disutility, Standard Demand = Discount Factor + Intratemporal Preference, Labor Supply = Labor Supply, Technology = Consumption Tech + Investment Tech + Common Tech

shocks consistently move in the same direciton as the Solow residual with one exception in the mid 1990s. The utilization-adjusted Solow residual, however, is entirely explained by technology shocks. This suggests that accounting for utilization adjustments, particularly in response to changes in demand, is important in correctly accounting for movements in the unadjusted Solow residual.

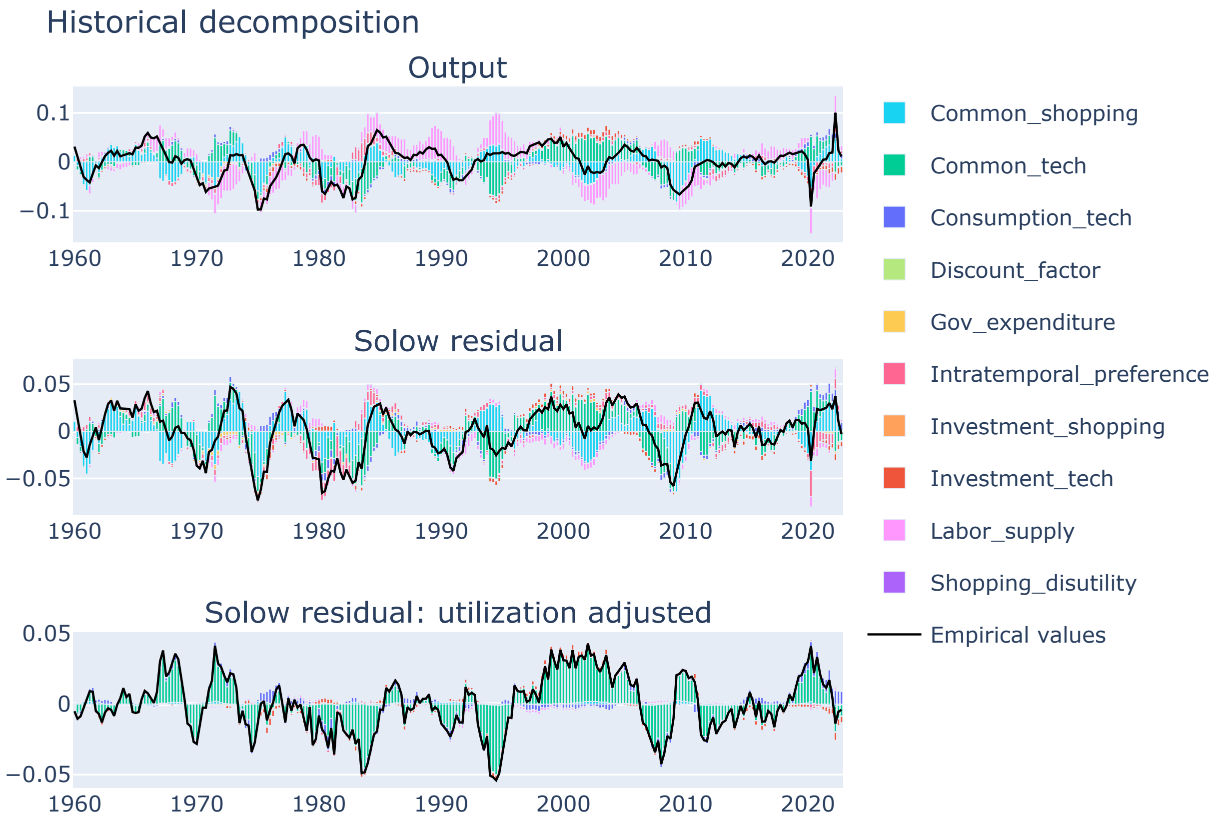

Figure 5: Historical decomposition across all 10 shock processes. Time period is 1960Q1-2022Q4. The solid black curve in each panel is the observable series filtered by the Hamilton method with values $p = 4$ and $h = 8$. Positive shocks are stacked on top of the horizontal axis, whereas negative shocks are stacked below.

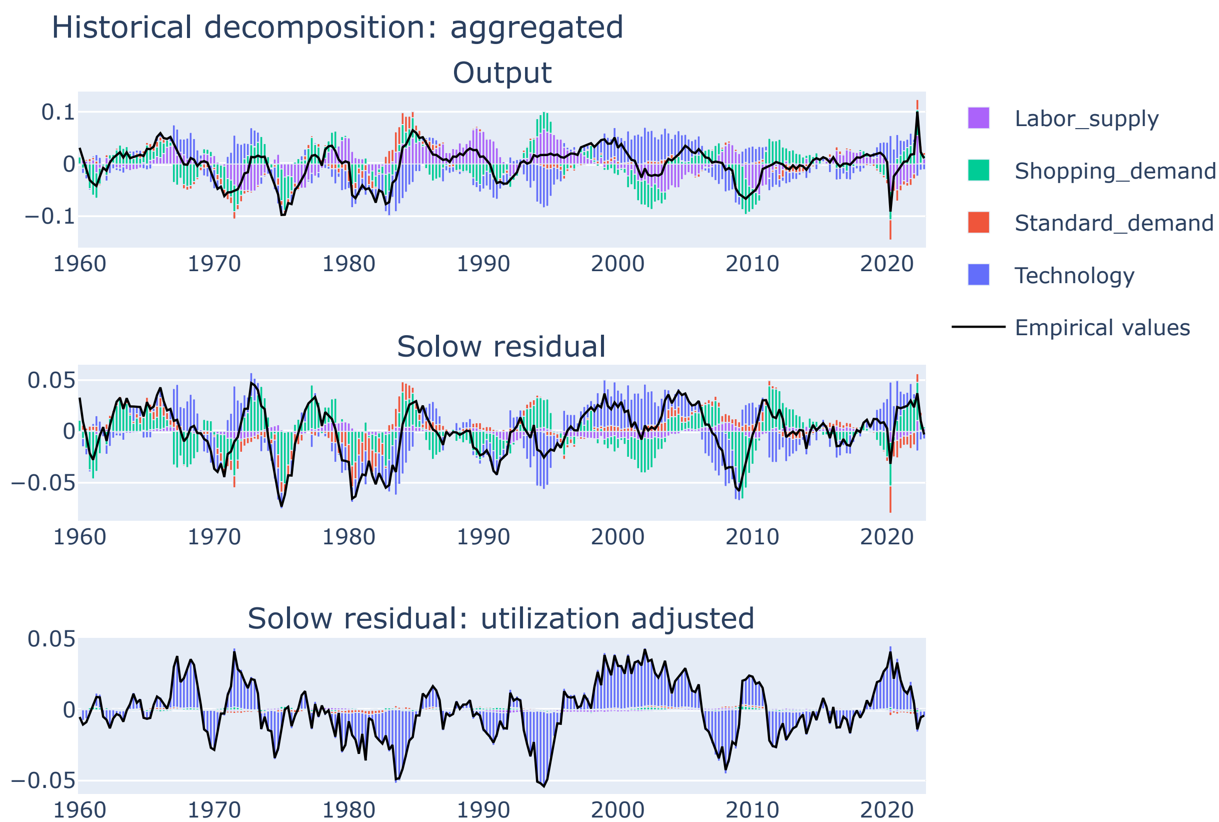

Figure 6: Historical decomposition across 4 categories of shocks: Shopping Demand = Common Shopping + Investment Shopping + Shopping Disutility, Standard Demand = Discount Factor + Intratemporal Preference, Labor Supply = Labor Supply, Technology = Consumption Tech + Investment Tech + Common Tech

4.6. Impulse responses

Figures 7 and 8 show the impulse response functions for 1 standard deviation shocks to common technology and shopping efficiency shocks, respectively. The impact on consumption, investment, labor supply, output, and the Solow residual from both shocks are similar qualitatively and in magnitude. Both output and the Solow residual rise by over 1 percent on impact, then gradually revert to their steady state values. The mechanism is, however, different. A production technology shocks directly raises the efficiency of production, reflected in the utilization-adjusted Solow residual rising by nearly 1 percent, in both the consumption and investment sector having a first-order effect on output. A shopping demand shock, on the other hand, has no significant effect on the adjusted Solow residual. Rather, it causes households to increase their shopping intensity, increasing matching with firms and input utilization.

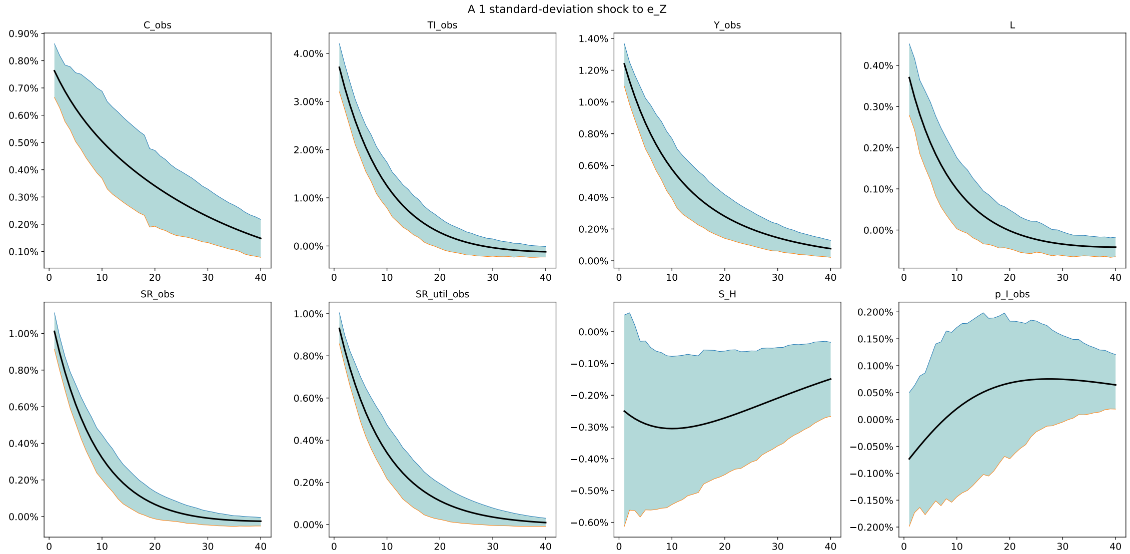

Figure 7: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

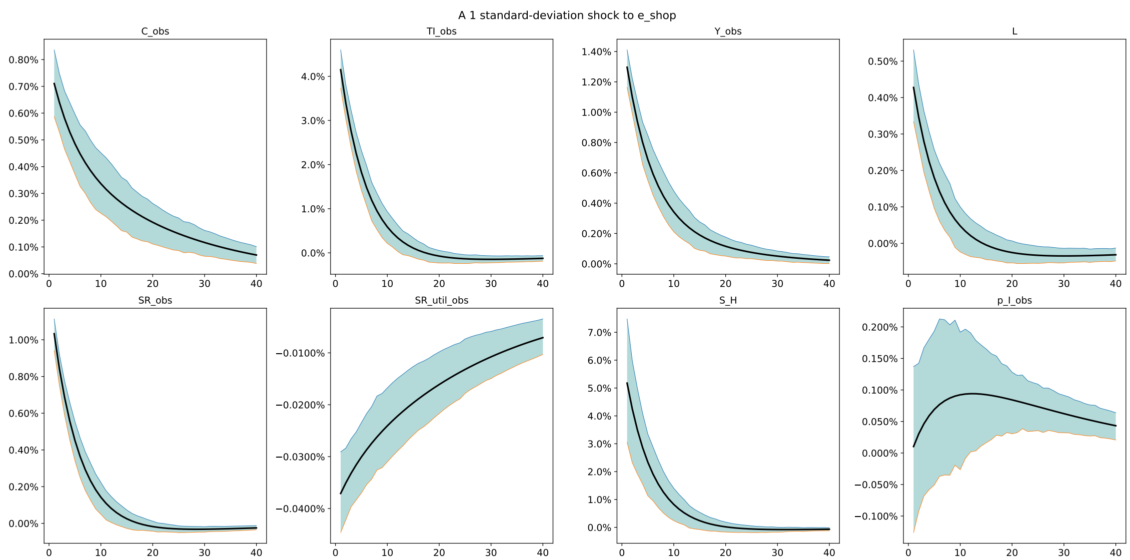

Figure 8: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

Figures 9,10, and 11 show the impulse response functions for 1 standard deviation shocks to investment shopping efficiency, investment production technology, and intratemporal preferences respectively. A positive shock to the investment sector, either through improved shopping efficiency or production technology, increases investment, output, and the Solow residual on impact. From intertemporal substitution, consumption initially falls to reallocate resources but then gradually rises as the rate of return to investment declines and the shock dies down. Whereas an investment-specific technology shock directly raises the utilization-adjusted Solow residual, the one to investment shopping efficiency operates by making it easier to acquire and bundle capital goods.

The intratemporal preference shock increases consumption on impact, and consequently crowds out investment, though the posterior bands are wide. Nevertheless, total output, labor supply and the Solow residual all increase. Households boost shopping effort alongside consumption, which causes the Solow residual to rise. The adjusted Solow residual diminishes by a negligible amount, reflecting composition bias but no change in technology. The rise in output can be interpreted in terms of both the increase in labor supply and boost in the Solow residual, which dominates the small decrease in the capital stock. Alternatively, the rise in consumption dominates the fall in investment, both because the response is percentage-wise much bigger and also because consumption is a larger share of output.

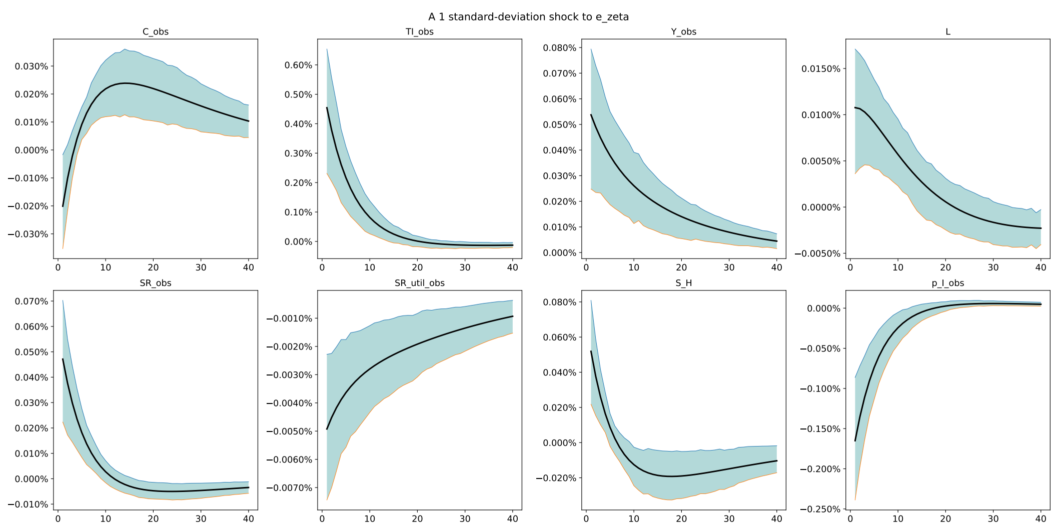

Figure 9: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

Figure 12 show the impulse response functions for 1 standard deviation shocks to government expenditure. Although increased government spends crowds out private consumption and investment, there is still a small increase in total output on impact.

In summary, the demand shocks increase search effort in the private consumption sector, output, and the Solow residual without significantly affecting the utilization-adjusted Solow residual. Hence, positive demand shocks can appear as technology enhancing if one uses the

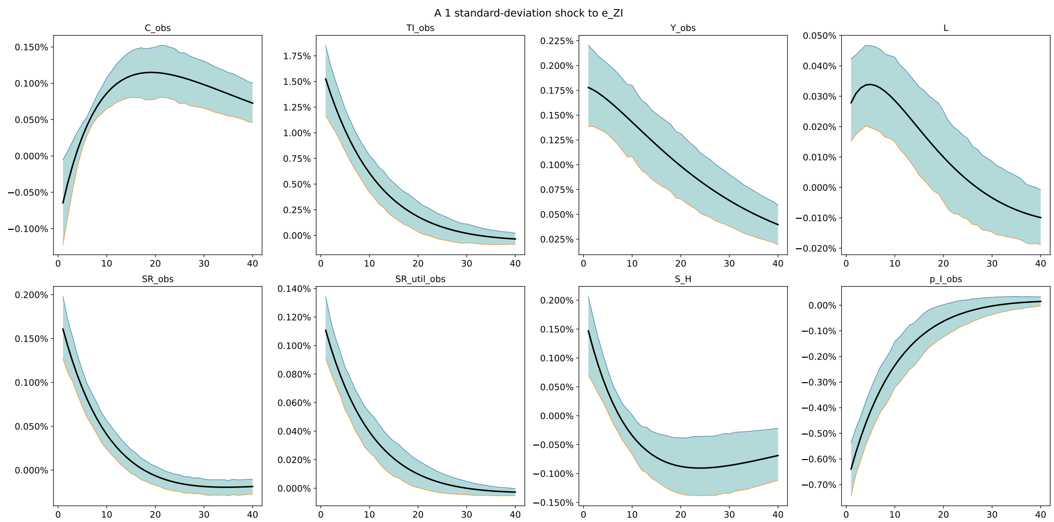

Figure 10: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

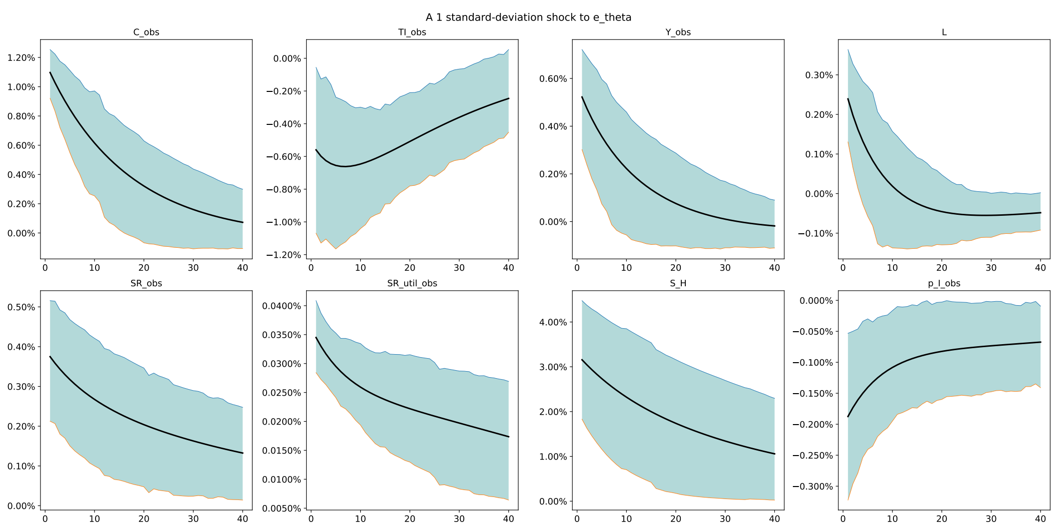

Figure 11: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

conventional measure of the Solow residual, when in reality there has been zero change in the production technology.

Note that the common technology and shopping shocks are unique in generating positive comovement among consumption, investment, labor supply, output, and the Solow residual. Intratemporal preference shocks, for instance, crowd out investment slightly, and shocks to the productivity of investment or investment shopping temporarily induce a fall in consumption due to intertemporal substitution. Thus, it is intuitive that these two series together

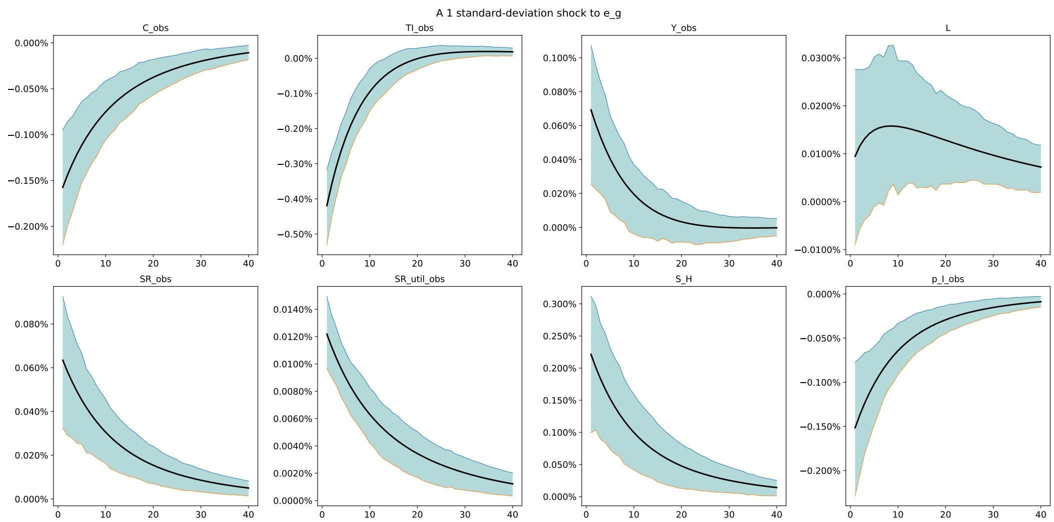

Figure 12: Mean posterior impulse respond in black line. Shaded region represents 90% highest posterior density interval.

dominate the forecast error variance decomposition of output and the Solow residual. However, general technology shocks induce a nearly identical comovement between the measured and adjusted Solow residuals, whereas the shopping shocks generate very different patterns. Since the time series' properties of the two series are quite different, common shopping demand shocks play an important role in the variance decomposition.

We have followed Bai, Rios-Rull, and Storesletten (2012) in keeping prices flexible. This choice simplifies the analysis and focuses attention on goods market frictions as a role for the transmission of demand shocks to productivity. However, under sticky prices, technology shocks tend to be contractionary in the short run, which accords with the empirical evidence found by Basu, Fernald, and Kimball (2006). Thus, generating the positive comovement between consumption, investment, labor supply, and output would require a greater short-run role for demand shocks. In effect, by assuming flexible prices, we have tilted the playing field in favor of technology shocks, and show that demand shocks nevertheless play an important role if one considers information on the adjusted Solow residual.

4.7. Estimation without including the adjusted Solow residual as an observable

To highlight the importance of using a utilization-adjusted measure of the Solow in the estimation, we re-estimate the model using only the standard Solow residual. Figure (13) shows the forecast error variance decomposition for all 10 shock processes and Figure (14) shows the decomposition where shocks are aggregated. Without using the utilization-adjusted

Solow residual, technology shocks explain a majority of variation of output and the Solow residual even in short time horizons. Moreover, both types of TFP are extremely correlated with each other and similarly correlated with output, contrary to the data. Shopping demand shocks no longer play a role in determining investment either.

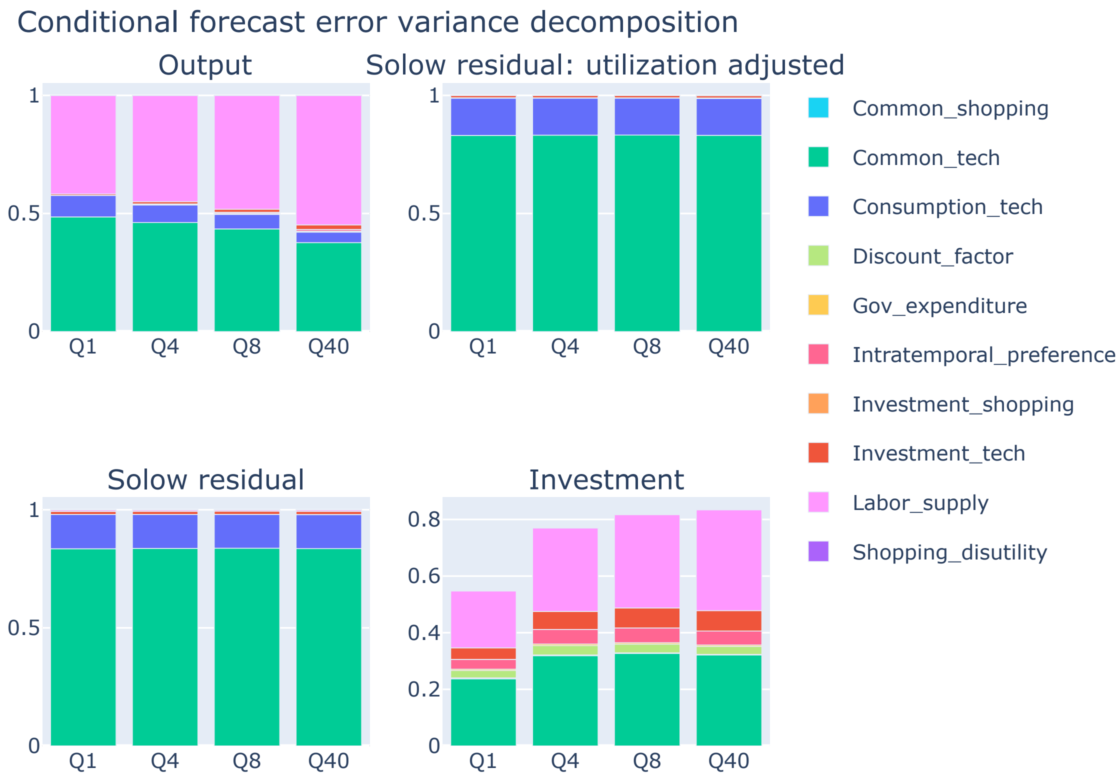

Figure 13: Forecast error variance decomposition. Model is estimated without utilization-adjusted Solow residual.

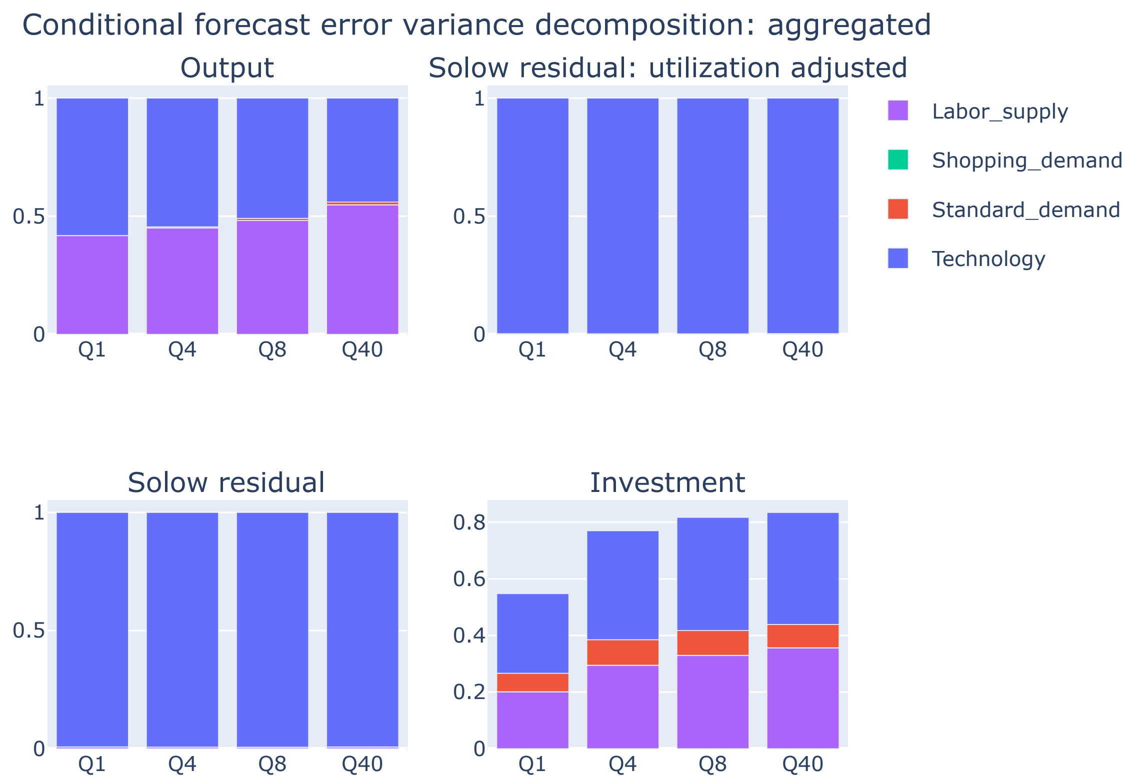

Figure 14: Forecast error variance decomposition across 4 categories of shocks: Shopping Demand = Common Shopping + Investment Shopping + Shopping Disutility, Standard Demand = Discount Factor + Intratemporal Preference, Labor Supply = Labor Supply, Technology = Consumption Tech + Investment Tech + Common Tech. Model is estimated without utilization-adjusted Solow residual.

Finally, if the model is estimated without using the utilization-adjusted Solow residual, then technology shocks explain a majority of variation of output and the Solow residual even in short horizons. Moreover, both types of TFP are extremely correlated with each other and similarly correlated with output, contrary to the data.

5. Conclusion

The utilization-adjusted TFP series leads TFP and is less correlated with output than the latter. We estimate a two-sector business cycle model featuring imperfect competition, variable capacity utilization, and shopping frictions on a rich set of observables including the standard Solow residual as well as a utilization-adjusted Solow residual. Including both Solow residual series provides distinct information about TFP that allow us to assess the importance of goods market frictions in shaping business cycles. We find that demand shocks play an important role in explaining observed variation in both output and total factor productivity. Both shopping and technology shocks generate positive comovement among consumption, investment, labor supply, output, and the Solow residuals. However, shopping shocks generate positive movements in the measured TFP without having a significant effect on the utilization-adjusted version, whereas with technology shocks the two series track each other closely. Intratemporal preference shocks, in turn, raise consumption, output, labor supply, and the Solow residual, but crowd out investment, which limits their role in contributing to business cycles. Including the utilization-adjusted Solow residual is essential for identifying the productive role of demand shocks.

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Appendix A. Data appendix

Table A.6 describes the raw data sources used in both the motivation and estimation. Each series used as an observable ranges from 1960Q1 to 2022Q4.

ID	Description	Source
LABSHPUSA156NRUG	Labor share of income	University of Groningen
PCND	Personal consumption: non-durable	BEA
PCESV	Personal consumption: services	BEA
HOANBS	Nonfarm business hours worked	BLS
CPIAUCSL	Consumer price index	BLS
GDPC1	Real GDP	BEA
GDPIC1	Real gross private domestic investment	BEA
CNP160V	Civilian non-institutional population	BLS

Table A.6: Data sources used in motivating evidence and estimation.

Appendix B. Additional derivations

Appendix B.1. Solution to household problem

Let λ_t be the Lagrangian multiplier on the budget constraint and μ_t be the multiplier on the composite term X_t. To simplify the exposition, let $\Gamma_t = c_t - \chi(L_t^{\psi}/\psi)X$. We also recognize that CES preferences over varieties implies for any given level of consumption c_t the demand curve for an individual variety is $(c_t/c_{i,t})^{1/\varepsilon} = \rho_{i,t}$.

The problem facing the household is to choose its level of consumption, shopping effort,and labor hours, and shares of equity. The first order conditions are

$$[c_t]: \quad b_t\theta_t\Gamma_t^{-\sigma} - P_t^H\lambda_t - \mu_t\gamma(X_{t-1}/c_t)^{1-\gamma} = 0 \tag{B.1}$$

$$[X_t]: \quad -b_t\theta_t\Gamma_t^{-\sigma}\chi\frac{L_t^{\psi}}{\psi} + \mu_t - \beta\mathbb{E}_t\mu_{t+1}(1-\gamma)c_{t+1}^{\gamma}X_t^{-\gamma} = 0 \tag{B.2}$$

$$[c_{it}]: \quad (b_t\theta_t\Gamma_t^{-\sigma} - \mu_t\gamma(X_{t-1}/c_t)^{1-\gamma})\frac{\partial c_t}{\partial c_{it}} - s_tA(Q_t^H)^{1-\phi}p_{it}\lambda_t = 0 \tag{B.3}$$

$$[s]: \quad (b_t\theta_t\Gamma_t^{-\sigma} - \mu_t\gamma(X_{t-1}/c_t)^{1-\gamma})\frac{\partial c_t}{\partial s_t} - b_t\kappa - \lambda_t\frac{\partial(P_tc_t)}{\partial s_t} = 0 \tag{B.4}$$

$$[L_t]: \quad -b_t\theta_t\Gamma_t^{-\sigma}\chi_tL_t^{\psi-1}X_t + \lambda_tw_t = 0$$

$$[x_{t+1}]: -\lambda_t\nu_t + \beta\mathbb{E}_t\lambda_{t+1}\{(\nu_{t+1} + d_{t+1})\} = 0 \tag{B.5}$$

Rearranging (B.2) yields

$$\mu_t = \frac{b_t\theta_t\Gamma_t^{-\sigma}\chi_t L_t^{\psi}}{\psi} + \beta\mathbb{E}_t\left\{\mu_{t+1}(1-\gamma)c_{t+1}^{\gamma}X_t^{-\gamma}\right\}$$

Combine (B.1) and (B.3) to find

$$\frac{\partial c_t}{\partial c_{it}} = s_t A(Q_t^h)^{1-\phi}\rho_{it}$$

To characterize the behavior of shopping and consumption we need two auxiliary expressions,

$$\begin{aligned}\frac{\partial c_t}{\partial s_t} &= \frac{\varepsilon}{\varepsilon-1}A(Q_t^H)^{1-\phi}\rho_{i,t}c_{it}\\ \frac{\partial(P_t c_t)}{\partial s_t} &= A(Q_t^H)^{1-\phi}p_{it}c_{it}\end{aligned}$$

which shows the marginal impact of shopping effort on the consumption basket and total expenditure respectively.

Equation (B.1) equates the marginal utility of market consumption to the marginal utility of wealth. Equation (B.4) says that the benefit of extra search equals the foregone leisure value. The benefit of extra search is the net utility from switching expenditure from existing goods to new goods. Equation (9) equates the wage to the marginal rate of substitution between consumption and leisure.

Substituting (B.1) into (B.4), and making use of the two auxiliary equations, obtains

$$\begin{aligned}b_t\kappa_t &= P_t^H\lambda_t\frac{\varepsilon}{\varepsilon-1}A(Q_t^H)^{1-\phi}\rho_{it}c_{it} - \lambda_t A(Q_t^H)^{1-\phi}p_{it}c_{it}\\ &= P_t^H\lambda_t\left(\frac{A(Q_t^H)^{1-\phi}\rho_{it}c_{it}}{\varepsilon-1}\right)\end{aligned}$$

Multiplying both sides by s_t we obtain

$$b_t s_t\kappa = P_t^H\lambda_t\frac{c_t}{\varepsilon-1} \tag{B.6}$$

where the last equality follows from evaluating the expression for total expenditure $P_t c_t = \int_0^{s_t A(Q_t^H)^{1-\phi}} p_{i,t}c_{i,t}di$ and using the demand curve for individuals varieties. Several insights can be obtained from equation (B.6). First, the value of shopping depends on an imperfect ability to substitute among goods, as $s \to 0$ as $\varepsilon \to \infty$. Second, the consumption level has opposing effects on shopping effort. Consumption expenditure is a prerequisite for shopping effort, but it also tends to reduce shopping effort through wealth effects. The force of this latter channel depends on the inverse intertemporal elasticity of substitution σ and the habit stock parameter h.

Equation (B.5) can be rearranged into a standard Euler equation for equity:

$$\nu_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} (\nu_{t+1} + d_{t+1}) \right\}$$

Appendix B.2. Solution to capital goods supplier's problem

To tackle this problem, first substitute (13) into the objective. Let η_t be the Lagrangian multiplier on (14). Then we use the following relations, analogous to the household problem:

$$\frac{\partial i_t}{\partial l_t} = \frac{\varepsilon_i}{\varepsilon_i - 1} \zeta A (Q_t^I)^{1-\phi} \rho_{jt}^I i_{jt} \tag{B.7}$$

$$\frac{\partial P_t^I i_t}{\partial l_t} = \zeta A (Q_t^I)^{1-\phi} p_{jt}^i i_{jt}$$

Let λ_t be the Lagrange multiplier on(13) and μ_t the multiplier on (14). The first-order conditions to the problem are:

$$[l_t]: \quad -w_t \lambda_t - \lambda_t \frac{\partial (P_t^I i_t)}{\partial l_t} + \eta_t \left(1 - \Phi' \left(\frac{i_t}{k_t} \right) \right) \frac{\partial i_t}{\partial l_t} = 0 \tag{B.8}$$

$$[i_t]: \quad -\lambda_t P_t^I + \eta_t \left(1 - \Phi' \left(\frac{i_t}{k_t} \right) \right) = 0 \tag{B.9}$$

$$[u_t^i]: \quad \lambda_t r_t^i k_t^i - \eta_t \delta^{i'}(u_t^i) k_t^i = 0, \quad i \in \{C, I\}$$

$$[k_{t+1}]: \quad -\eta_t + \beta \mathbb{E}_t \left\{ \lambda_{t+1} r_{t+1}^i u_{t+1}^i + \eta_{t+1}^i \left(1 - \delta^i(u_t^i) + \Phi^{i'} \left(\frac{i_{t+1}^i}{k_{t+1}^i} \right) \frac{i_{t+1}^i}{k_{t+1}^i} - \Phi^i \left(\frac{i_{t+1}}{k_{t+1}} \right) \right) \right\} = 0 \tag{B.10}$$

Using (B.9) in conjunction with the special relations (B.7) and (B.8) yields

$$\begin{aligned} w_t &= P_t^I \frac{\varepsilon_i}{\varepsilon_i - 1} \zeta A (Q_t^I)^{1-\phi} \rho_{jt}^I i_{jt} - \zeta A (Q_t^I)^{1-\phi} p_{jt}^I i_{jt} \\ &= \frac{\varepsilon_i}{\varepsilon_i - 1} \zeta A (Q_t^I)^{1-\phi} p_{jt}^I i_{jt} - \zeta A (Q_t^I)^{1-\phi} p_{jt}^I i_{jt} \\ &= \frac{1}{\varepsilon_i - 1} \zeta A (Q_t^I)^{1-\phi} p_{jt}^I i_{jt} \end{aligned}$$

Now, multiply both sides by l_{kt} and use (13) to find

$$w_t l_t = \frac{P_t^I i_t}{\varepsilon_i - 1}$$

Appendix B.3. Proof of lemma

We start with the equations for household and government consumption:

$$C = pZ^C(A(Q^h)^{-\phi})^{1-\alpha_2}(K^{Ch})^{\alpha}(L^{Ch})^{1-\alpha}\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$
$$G = pZ^C(A(Q^g)^{-\phi})^{1-\alpha_2}(K^{Cg})^{\alpha}(L^{Cg})^{1-\alpha}\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$$

Taking ratios yields

$$\begin{aligned}\frac{C}{G} &= \left(\frac{Q^h}{Q^g}\right)^{-\phi(1-\alpha_2)}\left(\frac{K^{Ch}}{K^{Cg}}\right)^{\alpha}\left(\frac{L^{Ch}}{L^{Cg}}\right)^{1-\alpha}\\ &= \left(\frac{Q^h}{Q^g}\right)^{-\phi(1-\alpha_2)}\frac{C}{G}\end{aligned}$$

using (25). Rearranging yields $Q^h = Q^g$.

Appendix C. Details on equilibrium

Appendix C.1. Equilibrium conditions (baseline)

Variety effect: household	$\frac{p_t}{P_t^H} = (A(S_t^H)^\phi (N_t^H)^{1-\phi})^{1/(\varepsilon-1)}$
Variety effect: investment	$\frac{1}{P_t^I} = (A(\zeta L_t^K)^\phi (N_t^I)^{1-\phi})^{1/(\varepsilon_i-1)}$
Profits: consumption	$d_t = \frac{Y_t^C}{\varepsilon N_t^C}$
Profits: investment	$d_t = \frac{I_t}{\varepsilon_i N_t^I}$
Labor intratemporal optimality	$\frac{w_t}{P_t^H} = \frac{\chi_t L_t^{1/\psi}}{\theta_t \Gamma_t^{-\sigma}}$
Capital accumulation	$K_{t+1} = (1-\delta^C(u_t^C))k_t^C + (1-\delta^I(u_t^I))k_t^I + i_t$
Household shopping	$\kappa_t S_t^h = \frac{\theta_t \Gamma_t^{-\sigma} c_t}{\varepsilon - 1}$
Consumption marginal utility	$P_t^h \lambda_t = b_t \theta_t \Gamma_t^{-\sigma}$
Utility component	$\Gamma_t = c_t - hc_{t-1}$
Euler equation (capital)	$q_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^i u_{t+1}^i + q_{t+1} \left(1 - \delta^i(u_t^i) + \Phi^{i'} \left(\frac{i_{t+1}}{k_{t+1}} \right) \frac{i_{t+1}}{k_{t+1}} - \Phi^i \left(\frac{i_{t+1}}{k_{t+1}} \right) \right) \right] \right\}$
Capital utilization	$r_t^i = \delta'^i(u_t^i), \quad i = \{C, I\}$
Rental rate (consumption)	$r_t^C = \frac{\alpha Y_t^C}{\frac{\varepsilon}{\varepsilon-1} u_t^C K_t^C}$
Rental rate (investment)	$r_t^I = \frac{\alpha I_t}{\frac{\varepsilon_i}{\varepsilon_i-1} u_t^I K_t^I}$
Free mobility	$\frac{N_t^H}{N_t^G} = \frac{C_t}{G_t}$
Tightness: consumption	$Q_t^C = N_t^H / S_t^H$
Tightness: investment	$Q_t^I = N_t^I / (\zeta L_t^K)$
Retail production	$Y_t^C = (A(Q_t^C)^{-\phi})^{1-\alpha_2} p Z_t^C (u_t^C K_t^C)^\alpha (L_t^C)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$
Investment	$I_t = (A(Q_t^I)^{-\phi})^{1-\alpha_2} Z_t^I (u_t^I K_t^I)^\alpha (L_t^I)^{1-\alpha} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}{(1-\alpha)^{1-\alpha}}$
Firm composition	$1 = N_t^H + N_t^G + N_t^I$
Aggregate accounting	$Y_t^C + I_t = w_t L_t + d_t + r_t^C u_t^C k_t^C + r_t^I u_t^I k_t^I$
Employment in investment shopping	$w_t L_t^K = \frac{I_t}{\varepsilon_i - 1}$
Capital composition	$K_t = K_t^C + K_t^I$
Labor composition	$L_t = L_t^C + L_t^I + L_t^K$

Appendix C.2. Log linearized system of baseline and no-entry models

Table C.7 describes the log linearized system for the baseline model.

Label	Equation
Variety effects: household	$(\varepsilon-1)(p-P^H)=\phi S+(1-\phi)N^H$
Variety effects: investment	$(\varepsilon_i-1)q=\phi(\zeta+L^K)+(1-\phi)N^I$
Consumption aggregation	$C=P^H+c$
Profits: consumption	$d=Y^C-N^C$
Profits: investment	$d=I-N^I$
Investment value	$w+L^I=I$
Factor prices	$w+L^C=Y^C$
Labor intratemporal	$b+\theta-\sigma\Gamma+\chi+X+(\psi-1)L=\lambda+w$
Capital accumulation	$K=(1-\delta^K)K_{-1}+\delta^K(I_{-1})$
Composite utility component	$\Gamma=\frac{\psi}{\psi-\overline{L}^\psi}c-\frac{\overline{L}^\psi}{\psi-\overline{L}^\psi}(\psi L+X+\chi)$
Household shopping effort	$S^H=P^H+\lambda+c-b-\kappa$
Consumption multiplier	$b+\theta-\sigma\Gamma=\left(1-\frac{(1+r)\gamma}{r+\gamma}\frac{\overline{L}^\psi}{\psi}\right)(P^H+\lambda)+\frac{(1+r)\gamma}{r+\gamma}\frac{\overline{L}^\psi}{\psi}(\mu+(1-\gamma)(X_{t-1}-c))$
Relative price of capital	$q=P^I+\Psi_K\delta^K(I-K)$
Variable X	$X=\gamma c+(1-\gamma)X_{-1}$
Euler equation (X)	$\mu=\frac{r+\gamma}{1+r}(b+\theta-\sigma\Gamma+X+\psi L)+\frac{1-\gamma}{1+r}\mathbb{E}(\mu'+\gamma(c'-X))$
Euler equation (capital)	$q=\mathbb{E}\{\lambda'-\lambda+\frac{r+\delta^K}{1+r}(r'+u'-u)+\frac{q}{1+r}\}$
Optimal utilization	$r^i=\Omega^i u^i \quad i\in\{C,I\}$
Rental rate (consumption)	$r^C=Y^C-u^C-K^C$
Rental rate (investment)	$r^I=I-u^I-K^I$
Tightness: consumption	$Q^C=N^H-S^H$
Tightness: investment	$Q^I=N^I-\zeta-L^K$
Free mobility	$N^H-N^G=Y^C-G$
Retail production	$Y^C=p+Z^C+(1-\alpha_2)(-\phi Q)+\alpha(u^C+K^C)+(1-\alpha)L^C$
Retail (utilization-adjusted)	$Y^{C,util}=p+Z^C+\alpha(K^C)+(1-\alpha)L^C$
Investment	$I=q+Z^I-\phi(1-\alpha_2)Q^I+\alpha(u^I+K^I)+(1-\alpha)L^I$
Investment (utilization-adjusted)	$I^{util}=Z^I+\alpha K^I+(1-\alpha)L^I$
Household consumption	$C=N^H-N^C+Y^C$
Composition of firms	$N=\phi^{NC}\left[(1/(1+g_c))N^H+(g_c/(1+g_c))N^G\right]+(1-\phi^{NC})N^I$
Aggregate expenditure	$Y=\frac{(r+\delta)\varepsilon[r+(1-\alpha)\delta^K]C+[\delta^K\alpha(\varepsilon-1)(r+\delta)+\delta(r+\delta^K)]TI}{(r+\delta)\varepsilon[r+(1-\alpha)\delta^K]+\delta^K\alpha(\varepsilon-1)(r+\delta)+\delta(r+\delta^K)}$
Aggregate income	$Y=\frac{(r+\delta^K)[(r+\delta)(\varepsilon-1)+\delta](w+L)+(r+\delta)(r+(1-\alpha)\delta^K)(N+d)}{(r+\delta)\varepsilon[r+(1-\alpha)\delta^K]+\delta^K\alpha(\varepsilon-1)(r+\delta)+\delta(r+\delta^K)}$
Total investment	$TI=\frac{\delta^K\alpha[(\varepsilon-1)(r+\delta)+\delta]I+\delta(r+(1-\alpha)\delta^K)(\nu+N_E)}{\delta(r+(1-\alpha)\delta^K)+\delta^K\alpha[(\varepsilon-1)(r+\delta)+\delta]}$
Employment in investment shopping	$w+L^K=I$
Labor composition	$L=(1-\phi^{LK})(\phi^C L^C+\phi^I L^I)+\phi^{LK}L^K$
Capital composition	$K=\phi^C K^C+\phi^I K^I$
Stochastic processes	$x=\rho_x x_{-1}+\varepsilon_x \quad \text{for} \quad x\in\{Z^C,Z^I,\theta,b,\kappa,\zeta,\chi\}$

Table C.7: Log linearized system of baseline model. The table omits the symbol $\tilde{}$, which denotes log deviations from steady state and abuse notation by using the equality sign $=$ rather than the approximation sign $\approx$

The system can be simplified as follows by eliminating L^I and K^I using the following relations:

$$w + L^I = I$$
$$w + L^I = Y^C - (u^C - u^I) - (K^C - K^I)$$

These yield the following simplified expressions for K^I:

$$K^I = I - Y^C + u^C - u^I + K^C$$

Label	Equation
Variety effects: household	$(\varepsilon - 1)(p - P^H) = \phi S + (1-\phi)N^H$
Variety effects: investment	$(\varepsilon_i - 1)q = \phi(\zeta + L^K) + (1-\phi)N^I$
Consumption aggregation	$C = P + c$
Profits: consumption	$d = Y^C - N^C$
Profits: investment	$d = I - N^I$
Labor in retail	$w + L^C = Y^C$
Capital accumulation	$K = (1-\delta_K)K_{-1} + \delta_K I_{-1}$
Composite utility component	$\Gamma = \frac{\psi}{\psi - \overline{L}^{\psi}} c - \frac{\overline{L}^{\psi}}{\psi - \overline{L}^{\psi}}(\psi L + X + \chi)$
Labor intratemporal	$b + \theta - \sigma\Gamma + \chi + X + (\psi - 1)L = \lambda + w$
Household shopping effort	$S^H = P^H + \lambda + c - b - \kappa$
Consumption multiplier	$b + \theta - \sigma\Gamma = \left(1 - \frac{(1+r)\gamma}{r+\gamma}\frac{\overline{L}^{\psi}}{\psi}\right)(P^H + \lambda) + \frac{(1+r)\gamma}{r+\gamma}\frac{\overline{L}^{\psi}}{\psi}(\mu + (1-\gamma)(X_{t-1} - c))$
Variable X	$X = \gamma c + (1-\gamma)X_{-1}$
Euler equation (X)	$\mu = \frac{r+\gamma}{1+r}(b + \theta - \sigma\Gamma + X + \psi L) + \frac{1-\gamma}{1+r}\mathbb{E}(\mu' + \gamma(c' - X))$
Relative price of capital	$q = P^I + \Psi_K \delta^K (I - K)$
Euler equation (capital)	$q = \mathbb{E}\{\lambda' - \lambda + \frac{r+\delta^K}{1+r}(r' + u' - u) + \frac{q}{1+r}\}$
Optimal utilization	$r^i = \Omega^i u^i \quad i \in \{C, I\}$
Rental rate (consumption)	$r^C = Y^C - u^C - K^C$
Rental rate (investment)	$r^I = I - u^I - K^I$
Free mobility	$N^H - N^G = C - G$
Tightness: consumption	$Q^C = N^H - S^H$
Tightness: investment	$Q^I = N^I - \zeta - L^K$
Tightness equalization	$N^H - S^H = N^G - S^G$
Retail production	$Y^C = p + Z^C + (1-\alpha_2)(-\phi Q) + \alpha(u^C + K^C) + (1-\alpha)L^C$
Retail (utilization-adjusted)	$Y^{C,util} = p + Z^C + \alpha K^C + (1-\alpha)L^C$
Investment	$I = Z^I - \phi(1-\alpha_2)Q^I + \alpha(u^I + K^I) + (1-\alpha)L^I$
Investment (utilization-adjusted)	$I^{util} = q + Z^I + \alpha K^I + (1-\alpha)L^I$
Composition of firms	$0 = \phi^{NC}\left[(1/(1+g_c))N^H + (g_c/(1+g_c))N^G\right] + (1-\phi^{NC})N^I$
Aggregate expenditure	$Y = Y_Y^C(Y^C) + I_Y(I)$
Aggregate income	$Y = \frac{(r+\delta^K)[(r+\delta)(\varepsilon-1)+\delta](w+L)+(r+\delta)(r+(1-\alpha)\delta^K)(N+d)}{(r+\delta)\varepsilon[r+(1-\alpha)\delta^K]+\delta^K\alpha(\varepsilon-1)(r+\delta)+\ \delta(r+\delta^K)}$
Employment in investment shopping	$w + L^K = I$
Labor composition	$L = (1-\phi^{LK})(\phi^C L^C + \phi^I L^I) + \phi^{LK} L^K$
Capital composition	$K = \phi^C K^C + \phi^I K^I$
Stochastic processes	$x = \rho_x x_{-1} + \varepsilon_x \quad \text{for} \quad x \in \{Z^C, Z^I, \theta, b, \chi, \kappa, \zeta, G\}$

Table C.8: Simplified log linear representation of the baseline model. The table omits the symbol ˜, which denotes log deviations from steady state and abuse notation by using the equality sign $=$ rather than the approximation sign $\approx$ for first-order approximations.

Appendix C.3. Key steady-state derivations

In this appendix, we compute all relevant steady-state ratios for the model. The Euler equation for capital implies that the steady real interest rate is

$$r = u^i r^i - \delta^i(u^i), \quad i = \{C, I\}$$

and steady state capital utilization must satisfy

$$r^i = \delta'^i(u^i) = \beta_1^i + \beta_2^i(u^i - 1), \quad i = \{C, I\}$$

We normalize the steady state utilization rate to unity which implies that

$$r^i = \beta_1^i = r + \delta^K, i \in \{C, I\}$$

We can let r^K denote the common rental rate in the steady state. Therefore β_1^i is determined by the rate of time preference and depreciation rate, while β_2^i is a free parameter, which will capture the sensitivity of utilization to the relative price of capital.

The law of motion for capital in the steady state requires $I = \delta^K K$. The shares of profits in the consumption and investment sectors is

$$\frac{N^C d}{Y^C} = \frac{1}{\varepsilon}, \quad \frac{N^I d}{I} = \frac{1}{\varepsilon_i}$$

The number of retail firms satisfies

$$N^C = \frac{Y^C}{Y^C + I\varepsilon/\varepsilon_i}$$

As expected, $N^C \to 1$ as $\varepsilon_i \to \infty$ and $\to 0$ as $\varepsilon \to \infty$. Moreover, the firm ratio coincides with the output shares if the markups are the same.

We calculate the share of investment to consumption. Optimal capital demand in each sector ensures

$$r^K K^I = \alpha I(\varepsilon_i/(\varepsilon_i - 1))^{-1}$$
$$r^K K^C = \alpha Y^C(\varepsilon/(\varepsilon - 1))^{-1}$$

$$r^K(K^C + K^I) = \alpha \left(\frac{Y^C}{\varepsilon/(\varepsilon - 1)} + \frac{I}{\varepsilon_i/(\varepsilon_i - 1)} \right)$$

Summing over the different types of capital, we have

$$r^K K = \alpha \left(\frac{Y^C}{\varepsilon/(\varepsilon-1)} + \frac{I}{\varepsilon_i/(\varepsilon_i-1)} \right) \tag{C.1}$$

Using the steady state condition for capital and utilization,

$$I = \delta^K K = \frac{\delta^K}{r^K}\alpha \left(\frac{Y^C}{\varepsilon/(\varepsilon-1)} + \frac{I}{\varepsilon_i/(\varepsilon_i-1)} \right)$$

and using the steady state extensive margin of investment to retail output we obtain

$$\frac{I}{Y^C} = \frac{\delta^K \alpha \left(\frac{\varepsilon-1}{\varepsilon}\right)}{r^K - \delta^K \alpha \left(\frac{\varepsilon_i-1}{\varepsilon_i}\right)}$$

The ratio of investment to retail output rises with the depreciation rate δ^K and output elasticity α, falls with the interest rate r, and falls with the retail markup and rises with the investment markup. Under perfect competition ($\varepsilon \to \infty, \varepsilon_i \to \infty$), the entry sector vanishes, and $I/Y^C \to \delta^K \alpha/(r^K - \delta^K \alpha)$.

It immediately follows that the ratio of capital to retail output is

$$\frac{K}{Y^C} = \frac{\alpha \left(\frac{\varepsilon-1}{\varepsilon}\right)}{r^K - \delta^K \alpha \left(\frac{\varepsilon_i-1}{\varepsilon_i}\right)} \tag{C.2}$$

Additionally, we can find the mass of consumption and investment firms directly in terms of parameters:

$$N^C = \frac{r^K \varepsilon_i - \delta^K \alpha(\varepsilon_i - 1)}{r^K \varepsilon_i + \delta^K \alpha(\varepsilon - \varepsilon_i)}$$
$$N^I = \frac{\alpha \delta^K (\varepsilon - 1)}{r^K \varepsilon_i + \delta^K \alpha(\varepsilon - \varepsilon_i)}$$

We now calculate the share of each sector's capital. Combining the steady state interest rate, payments to capital, and capital stock we have

$$\phi^I \equiv \frac{K^I}{K} = \frac{\varepsilon_i - 1}{\varepsilon_i} \frac{\alpha \delta^K}{r + \delta^K}$$

Similarly,

$$\begin{aligned} \phi^C \equiv \frac{K^C}{K} &= \frac{\varepsilon - 1}{\varepsilon} \frac{\alpha}{r^K} \frac{Y^C}{K} \\ &= \frac{\varepsilon_i r^K - \alpha \delta^K (\varepsilon_i - 1)}{\varepsilon_i (r + \delta^K)} \end{aligned}$$

We can now calculate the investment share of output $I_Y \equiv I/Y$

$$
\begin{aligned}
I_Y &= \frac{1}{1 + 1/(I/Y^C)} \\
&= \frac{\alpha\delta^K \varepsilon_i(\varepsilon - 1)}{r\varepsilon\varepsilon_i + \alpha\delta^K\varepsilon - \alpha\delta^K\varepsilon_i + \delta^K\varepsilon\varepsilon_i}
\end{aligned}
$$

From this, we find the output ratios of total investment and retail $Y_Y^C = Y^C/Y$:

$$
Y_Y^C = 1 - \frac{I}{Y}
$$

The share of household consumption $C_Y \equiv C/Y$

$$
C_Y = \frac{1}{1+g_c}\frac{Y^C}{Y}
$$

Aggregate profits equal d given the fact that there is a unit mass of firms.

$$
\begin{aligned}
\frac{d}{Y} &= \frac{N^C d + N^I d}{Y} \\
&= \frac{Y^C}{Y}\frac{1}{\varepsilon} + \frac{I}{Y}\frac{1}{\varepsilon_i} \\
&= \frac{r^K\varepsilon_i + \alpha\delta^K(\varepsilon - \varepsilon_i)}{r^K\varepsilon\varepsilon_i + \alpha\delta^K(\varepsilon - \varepsilon_i)}
\end{aligned}
$$

If we let $\delta^K \to 0$, then output coincides with retail sales, and the profit share approaches $1/\varepsilon$. Moreover, if $\varepsilon = \varepsilon_i$, then the consumption and investment sectors have an overall profit share of $1/\varepsilon$.

The income approach to output implies $Y = wL + r^K(u^C K^C + u^I K^I) + Nd$, so that the joint share of rental and labor income to output is therefore $1 - d/Y$.

Total labor income satisfies

$$
wL = (1-\alpha)\left(\frac{\varepsilon - 1}{\varepsilon}Y^C + \frac{\varepsilon_i - 1}{\varepsilon_i}I\right) + \frac{I}{\varepsilon_i - 1} \tag{C.3}
$$

Combining (C.3) with (C.1) yields so that

$$
\frac{wL}{r^K K} = \frac{1-\alpha}{\alpha} + \frac{I}{r^K K(\varepsilon_i - 1)}
$$

which in the steady state satisfies

$$
\frac{wL}{r^K K} = \frac{1-\alpha}{\alpha} + \frac{\delta^K}{(r+\delta^K)(\varepsilon_i - 1)}
$$

The ratio of wage to total factor income is

$$\frac{wL}{wL + r^K K} = \frac{(1-\alpha)(r+\delta^K)(\varepsilon_i - 1) + \delta^K \alpha}{(r+\delta^K)(\varepsilon_i - 1) + \delta^K \alpha}$$

The labor share in the investment shopping sector is

$$\phi^{LK} \equiv \frac{L^K}{L} = \frac{\alpha\delta^K}{\alpha\delta^K + (1-\alpha)(r+\delta^K)(\varepsilon_i - 1)}$$

Thus, the labor shares in consumption and investment satisfy

$$\frac{L^C}{L} = \phi^C(1-\phi^{LK}), \quad \frac{L^I}{L} = \phi^I(1-\phi^{LK}),$$

Within the consumption, investment, and entry sectors, the capital-labor ratio is the same. The steady-state labor share of income satisfies

$$\begin{aligned} \omega \equiv \frac{wL}{Y} &= \frac{wL}{wL + r^K K}\frac{wL + r^K K}{Y} \\ &= (1-\alpha)\left(1 - \frac{d}{Y}\right) \\ &= \frac{r^K(\varepsilon - 1)}{\frac{\alpha\delta^K(\varepsilon - \varepsilon_i)}{\varepsilon_i} + r^K\varepsilon} \frac{\frac{\alpha\delta^K}{r^K(\varepsilon_i - 1)} + 1 - \alpha}{\frac{\alpha\delta^K}{r^K(\varepsilon_i - 1)} + 1} \end{aligned}$$

Several special cases are worth considering. First, consider $\varepsilon_i \to \infty$. Then the labor share approaches

$$(1-\alpha)\frac{r^K(\varepsilon - 1)}{r^K\varepsilon - \alpha\delta^K}$$

No labor is employed in shopping for investment goods, so the share $1-\alpha$ is deflated by the average markup between the two sectors. In particular, as $\delta^K \to 0$, the labor share approaches $(1-\alpha)(\varepsilon - 1)/\varepsilon$. If both $\varepsilon, \varepsilon_i \to \infty$, then of course the labor share approaches $1-\alpha$. Finally, if $\varepsilon = \varepsilon_i$ at a finite value, then the labor share is both deflated by the average markup but also inflated by the role of workers shopping for investment goods.

We can turn to steady-state relations on the household side. We can normalize $b = \theta = \chi = 1$ in the steady state. We also use $c = X$. From the definition of Γ, we have

$$\Gamma = c\left(1 - \frac{L^\psi}{\psi}\right)$$

The consumption first order condition implies

$$\Gamma^{-\sigma} = P^H\lambda + \mu\gamma$$

The Euler for X in steady state becomes

$$\mu\left(\frac{r+\gamma}{1+r}\right)=\frac{L^{\psi}}{\psi}\left(P^{H}\lambda+\mu\gamma\right)$$

We can solve for labor supply. The first order condition for labor supply in the steady state yields

$$\Gamma^{-\sigma}L^{\psi}X=\lambda wL$$

Using $C=P^{H}c$, we have

$$\frac{wL}{C}=\frac{\Gamma^{-\sigma}L^{\psi}}{\lambda P^{H}}$$

We can in turn write wL/C directly in terms of parameters:

$$\begin{aligned}\frac{wL}{C}&=(1+g_c)\frac{wL}{Y^C}\\&=(1+g_c)\frac{wL}{r^K K}\frac{r^K K}{Y^C}\\&=(1+g_c)\left[\frac{1-\alpha}{\alpha}+\frac{\delta^K}{(r+\delta^K)(\varepsilon_i-1)}\right]\frac{(r+\delta^K)\alpha^{\frac{\varepsilon-1}{\varepsilon}}}{r+\delta^K-\delta^K\alpha^{\frac{\varepsilon_i-1}{\varepsilon_i}}}\end{aligned}$$

Now it remains to simplify $\Gamma^{-\sigma}L^{\psi}/(\lambda P^{H})$. Using the Euler for X, we obtain

$$\frac{\Gamma^{-\sigma}L^{\psi}}{\lambda P^{H}}=\frac{\psi\mu}{\lambda P^{H}}\left(\frac{r+\gamma}{1+r}\right)$$

We can further simplify:

$$\frac{\psi\mu}{\lambda P^{H}}\left(\frac{r+\gamma}{1+r}\right)=L^{\psi}\left(1+\frac{\mu\gamma}{\lambda P^{H}}\right)$$

so that

$$\frac{\mu}{\lambda P^{H}}=\frac{L^{\psi}}{\psi\frac{r+\gamma}{1+r}-L^{\psi}\gamma}$$

and hence

$$\frac{wL}{C}=\frac{\psi\frac{r+\gamma}{1+r}L^{\psi}}{\psi\frac{r+\gamma}{1+r}-L^{\psi}\gamma}$$

It follows that

$$L=\left(\frac{\psi\frac{r+\gamma}{1+r}\frac{wL}{C}}{\psi\frac{r+\gamma}{1+r}+\gamma\frac{wL}{C}}\right)^{1/\psi}$$

This part not updated!

Appendix C.4. Sequential computation of steady state

We can obtain the wage directly from (??) Now, given a guess for C and P^c, we immediately recover a consumption bundle $c = C/P^c$ as well as Γ. Government purchases satisfy $G = g_c C$. Hence, retail output satisfies $Y^C = C + G$. Household shopping effort can be obtained from the first order condition:

$$S^c = \frac{\theta \Gamma^{-\sigma} c}{\kappa(\varepsilon - 1)}$$

K can be pinned down using (C.2). Given K, investment in physical capital satisfies $I = \delta_K K$, and L can be obtained from the capital-labor ratio (??). Given L and K, we immediately recovery K^i and L^i for $i \in \{C, I, E\}$. The number of entrants satisfies $N_E = Z^E (K^E)^\alpha (L^E)^{1-\alpha}$, and we recover the total stock of firms from $N = (1-\delta)/\delta N_E$. The ratio of firms producing for households and the government satisfies $N^g/N^c = g_c$. Thus, $N^g = (g_c/(1+g_c))N$ and $N^c = (1/(1+g_c))N$.

The firm value ν satisfies $\nu = (Y^C/N)\delta/[(r+\delta)\varepsilon]$.

Given N^c and S^c, we calculate household tightness $Q^c = N^c/S^c$. We can check our guesses for C and P by using discrepancies in the retail production function and labor supply:

$$\mathcal{L}_1 = C - pZ(AQ^{-\phi})^{1-\alpha_2} \frac{(1-\alpha)^{1-\alpha}}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}}$$
$$\mathcal{L}_2 = L^{1/\psi} - \frac{\theta \Gamma^{-\sigma}}{\chi P} \frac{1-\alpha}{\alpha} (r + \delta^K) \left(\frac{Z^I \alpha}{r + \delta^K} \right)^{1/(1-\alpha)}$$

This procedure implicitly defines a loss function: $\mathcal{L}(C, P) : \mathbb{R}^{2+} \to \mathbb{R}^2$.

In the case $\sigma = 1$, the income and substitution effects cancel out, and we can characterize the steady state in closed form. It is most straightforward to solve for labor supply.

While (??) does not have a closed form expression, one is available under logarithmic preferences: $\sigma = 1$. Then, note that $\Gamma P = c(1-h)P = C(1-h)$. Using this, and multiplying both sides by K we get

$$L^{1+1/\psi} \frac{K}{L} = \frac{\theta}{\chi} \frac{1-\alpha}{\alpha} \frac{K}{C} \frac{r + \delta^K}{1-h} \frac{K}{L}$$

Canceling out K/L and replacing K/C, we obtain

$$L = \left(\frac{\theta}{\chi} \frac{1-\alpha}{r + (1-\alpha)\delta^K} \frac{r + \delta^K}{1-h} \left(\frac{\varepsilon - 1}{\varepsilon} + \frac{\delta}{(r+\delta)\varepsilon} \right) (1 + g_c) \right)^{1/(1+1/\psi)}$$

Together with the closed form expression for the wage, we obtain the capital stock:

$$K = \frac{\alpha}{1-\alpha}\frac{wL}{r+\delta^K}$$

Shopping effort simplifies to

$$S = \frac{\theta}{\kappa(1-h)(\varepsilon-1)}$$

Remaining quantities follow easily by using the ratios in Appendix C.3.

Appendix C.5. Derivation of select log linearized equations

1. Utility component Γ_t

   In levels, we have $\Gamma_t = c_t - \chi_t(L_t^\psi/\psi)X_t$. Applying a first-order Taylor series expansion yields

   $$\overline{\Gamma}\tilde{\Gamma}_t = \overline{c}\tilde{c}_t - \frac{\overline{\chi}\overline{L}^\psi\overline{X}}{\psi}(\psi\tilde{L}_t + \tilde{X}_t + \tilde{\chi}_t)$$

   Then using $\overline{c} = \overline{X}$, we have

   $$\tilde{\Gamma}_t = \frac{\psi\tilde{c}_t}{\psi - \overline{\chi}\overline{L}^\psi} - \frac{\overline{\chi}\overline{L}^\psi}{\psi - \overline{\chi}\overline{L}^\psi}(\psi\tilde{L}_t + \tilde{X}_t + \tilde{\chi}_t)$$

2. Variable X_t

   $X_t = c_t^\gamma X_t^{1-\gamma}$, so that

   $$\tilde{X}_t = \gamma\tilde{c}_t + (1-\gamma)\tilde{X}_{t-1}$$

3. Euler equation for X In levels, we have

   $$\mu_t = \frac{b_t\theta_t\Gamma_t^{-\sigma}\chi_t L_t^\psi}{\psi} + \beta\mathbb{E}_t\{\mu_{t+1}(1-\gamma)c_{t+1}^\gamma x_t^{-\gamma}\}$$

   Log linearizing, simplifying and dividing by $\overline{u}$ yields

   $$\tilde{\mu}_t = \frac{\overline{\Gamma}^{-\sigma}\overline{L}^\psi}{\psi\overline{\mu}}(\tilde{b}_t + \tilde{\theta}_t - \sigma\tilde{\Gamma}_t + \tilde{\chi}_t + \psi\tilde{L}_t) + \beta(1-\gamma)\gamma\mathbb{E}_t(\tilde{\mu}_{t+1} + \tilde{c}_{t+1} - \tilde{x}_t)$$

   Finally, from the steady state, we note that $\frac{\overline{\Gamma}^{-\sigma}\overline{L}^\psi}{\psi\overline{\mu}} = (\rho+\gamma)/(1+\rho)$, so that

   $$\tilde{\mu}_t = \frac{\rho+\gamma}{1+\rho}(\tilde{b}_t + \tilde{\theta}_t - \sigma\tilde{\Gamma}_t + \tilde{\chi}_t + \psi\tilde{L}_t) + \beta(1-\gamma)\gamma\mathbb{E}_t(\tilde{\mu}_{t+1} + \tilde{c}_{t+1} - \tilde{x}_t)$$

4. Consumption marginal utility λ_t

$$P_t\lambda_t = b_t\theta_t\Gamma_t^{-\sigma}$$
$$\tilde{P}_t + \tilde{\lambda}_t = \tilde{b}_t + \tilde{\theta}_t - \sigma\tilde{\Gamma}_t$$

5. Relative price of capital

   Here we use the normalization $P_t = 1$.

$$q_t\left(1 - \Phi_K'\left(\frac{I_t}{K_t}\right)\right) = 1$$
$$q_t\left(1 - \Psi_K\left(\frac{I_t}{K_t} - \delta^k\right)\right) = 1$$
$$\overline{q}e^{\tilde{q}_t}\left(1 - \Psi_K\left(\delta^K e^{\tilde{I}_t - \tilde{K}_t} - \delta^K\right)\right) = 1$$
$$\overline{q}e^{\tilde{q}_t} - \Psi_K\delta^K e^{\tilde{I}_t - \tilde{K}_t + \tilde{q}_t} + \Psi_K\delta^K e^{\tilde{q}_t - \tilde{P}_t^I} = 1$$
$$\tilde{q}_t = \Psi_K\delta^K(\tilde{I} - \tilde{K})$$

6. Depreciation function

   Let us also turn to depreciation as a function of utilization, omitting the i index for simplicity:

$$\delta(u_t) = \delta^E + \beta_1(u_t - 1) + \frac{\beta_2}{2}(u_t - 1)^2$$

   Hence, in log deviations we have

$$\delta^K e^{\hat{\delta}_t} = \delta^K + \beta_1(\overline{u}e^{\tilde{u}_t} - 1) + \frac{\beta_2}{2}(\overline{u}e^{\tilde{u}_t} - 1)^2$$

   Using $\overline{u} = 1$ and applying the first-order approximation, we have

$$\delta^K(1 + \tilde{\delta}_t) = \delta^K + \beta_1\tilde{u}_t + \frac{\beta}{2}\tilde{u}_t^2$$
$$\delta^E\tilde{\delta}_t = \beta_1\tilde{u}_t$$

   Now, using $\beta_1 = r + \delta^K$, we have

$$\tilde{\delta}_t = \frac{r + \delta^K}{\delta^K}\tilde{u}_t$$

7. Euler equation for capital

It's useful to simplify the adjustment cost term. Note that

$$\begin{aligned}
\Phi_K'\left(\frac{I}{K}\right)\frac{I}{K} &= \Psi_K\left(\frac{I}{K} - \delta^K\right)\frac{I}{K} \\
&= \Psi_K\delta^K\left(e^{\tilde{I}-\tilde{K}} - 1\right)\delta^K e^{\tilde{I}-\tilde{K}} \\
&= \Psi_K(\delta^K)^2\left(e^{2(\tilde{I}-\tilde{K})} - e^{\tilde{I}-\tilde{K}}\right) \\
&= \Psi_K(\delta^K)^2(\tilde{I}-\tilde{K}) \\
&= \delta^K\tilde{q}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Phi_K\left(\frac{I}{K}\right) &= \frac{\Psi_K}{2}\left(\frac{I}{K} - \delta^K\right)^2 \\
&= \frac{\Psi_K}{2}\left(\delta^K e^{\tilde{I}-\tilde{K}} - \delta^K\right)^2 \\
&= \frac{\Psi_K}{2}(\delta^K)^2\left(\tilde{I}-\tilde{K}\right)^2 \\
&= 0
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
\overline{q}e^{\tilde{q}_t} &= \beta\mathbb{E}_t\left\{e^{\tilde{\lambda}_{t+1}-\tilde{\lambda}_t}[(r+\delta^K)e^{\tilde{r}^K_{t+1}+\tilde{u}^i_{t+1}} + \overline{q}e^{\tilde{q}_{t+1}}(1-\delta^K e^{\tilde{\delta}^K_t} + \delta^K\tilde{q}_{t+1})]\right\} \\
\overline{q}e^{\tilde{q}_t} &= \beta\mathbb{E}_t\left\{(r+\delta^K)e^{\tilde{\lambda}_{t+1}-\tilde{\lambda}_t+\tilde{r}^K_{t+1}+u^i_{t+1}} + (1-\delta^K e^{\tilde{\delta}^K_t} + \delta^K\tilde{q}_{t+1})\overline{q}e^{\tilde{\lambda}_{t+1}-\tilde{\lambda}_t+\tilde{q}_{t+1}}\right\} \\
(1+r)\tilde{q}_t &= \mathbb{E}_t\left\{(1+r)(\tilde{\lambda}_{t+1}-\tilde{\lambda}_t + (r+\delta^K)(\tilde{r}^K_{t+1}+\tilde{u}^i_{t+1}) + \tilde{q}_{t+1} - \delta^K\tilde{\delta}^K_t\right\} \\
(1+r)\tilde{q}_t &= \mathbb{E}_t\left\{(1+r)(\tilde{\lambda}_{t+1}-\tilde{\lambda}_t + (r+\delta^K)(\tilde{r}^K_{t+1}+\tilde{u}_{t+1}-\tilde{u}_t) + \tilde{q}_{t+1}\right\} \\
\tilde{q}_t &= \mathbb{E}_t\left\{(\tilde{\lambda}_{t+1}-\tilde{\lambda}_t) + \frac{r+\delta^K}{1+r}(\tilde{Y}^C_{t+1} - \tilde{K}^C_{t+1} - \tilde{u}_t) + \frac{\tilde{q}_{t+1}}{1+r}\right\}
\end{aligned}$$

8. Utilization

   The utilization equation is

$$q_t\delta^{i'}(u_t) = r^K_{it}, \quad i \in \{C, I\}$$

   The quadratic relationship between utilization and depreciation implies that

$$\delta^{i'}(u) = \beta_1 + \beta_2^i(u_t - 1)$$

We rewrite optimal utilization as

$$\bar{q}e^{\tilde{q}_t^i}(\beta_1 + \beta_2^i(\bar{u}e^{\tilde{u}_t} - 1)) = \overline{r}^K e^{\tilde{r}_{i,t}^K}, \quad i = \{C, I\}$$

$$\beta_1 + \beta_2^i(\overline{u}e^{\tilde{u}_t} - 1) = \overline{r}^K e^{\tilde{r}_{i,t}^K - \tilde{q}_t}$$

$$\beta_1 + \beta_2^i \tilde{u}_t = \overline{r}(1 + \tilde{r}_{i,t}^K - \tilde{q}_t)$$

Now we use $\overline{r}^K = \beta_1 = (r + \delta^K)$ to simplify:

$$\beta_2^i \tilde{u}_t = (r + \delta^K)(\tilde{r}_{it}^K - \tilde{q}_t)$$

so that

$$\tilde{r}_{it}^K - \tilde{q}_t = \Omega^i \tilde{u}_t$$

where the composite parameter Ω^i satisfies $\Omega^i = \beta_2^i/(r + \delta^K)$

9. Aggregate income

   We decompose aggregate income and apply steady-state ratios:

$$Y_t = N_t d_t + w_t L_t + r_t^K (u^C K_t^C + u^I K_t^I + K_t^E)$$

$$\tilde{Y}_t = \frac{\overline{Nd}}{\overline{Y}}(\tilde{N}_t + \tilde{d}_t) + \frac{\overline{wL}}{\overline{Y}}(\tilde{w_t L_t}) + \frac{\overline{r^K K}}{\overline{Y}} \tilde{r_t^K}(\tilde{u_t^C K_t^C} + \tilde{u_t^I K_t^I} + \tilde{K_t^E})$$

$$\tilde{Y}_t = \frac{(r + \delta^K)[(r + \delta)(\varepsilon - 1) + \delta][(1 - \alpha)(\tilde{w}_t + \tilde{L}_t) + \alpha(\tilde{K}_t + \tilde{r}_t^K)] + (r + \delta)(r + (1 - \alpha)\delta^K)(\tilde{N}_t + \tilde{d}_t)}{(r + \delta)\varepsilon[r + (1 - \alpha)\delta^K] + \delta^K \alpha(\varepsilon - 1)(r + \delta) + \ \delta(r + \delta^K)}$$

$$\tilde{Y}_t = \frac{(r + \delta^K)[(r + \delta)(\varepsilon - 1) + \delta](\tilde{w}_t + \tilde{L}_t) + (r + \delta)(r + (1 - \alpha)\delta^K)(\tilde{N}_t + \tilde{d}_t)}{(r + \delta)\varepsilon[r + (1 - \alpha)\delta^K] + \delta^K \alpha(\varepsilon - 1)(r + \delta) + \ \delta(r + \delta^K)}$$

   where we apply $\tilde{r}_t^K + \tilde{K}_t = \tilde{w}_t + \tilde{L}_t$.