What Bell actually proofs

T. Liffert

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Abstract

The proof of John Stewart Bell implies, that three photons cannot be entangled. This matches with the fact, that, if one entangles a third photon with one of an entangled pair, the original entanglement ends and the state of the third photon is in 25 percent of the cases copied to one of the proeviously entangled ones.

1 Three photons

Imagine the following situation, where three λ-identical photons go to three polarizers, set to 0°, 30° and 60°:

- 0° channel: $\text{photon}_1 = (\lambda) \rightarrow$ 0°-polarizer
- 30° channel: $\text{photon}_2 = (\lambda) \rightarrow$ 30°-polarizer
- 60° channel: $\text{photon}_3 = (\lambda) \rightarrow$ 60°-polarizer

Let us denote the outcome at 0° with a, at 30° with b and at 60° with c. Let us consider the four cases

- $(a = b) \wedge (b = c) \implies (a = c)\,,\ \frac{3}{4} \cdot \frac{3}{4}$
- $(a = b) \wedge (b \neq c) \implies (a \neq c)\,,\ \frac{3}{4} \cdot \frac{1}{4}$
- $(a \neq b) \wedge (b = c) \implies (a \neq c)\,,\ \frac{1}{4} \cdot \frac{3}{4}$
- $(a \neq b) \wedge (b \neq c) \implies (a = c)\,,\ \frac{1}{4} \cdot \frac{1}{4}$

We get then the following weights for each 2-set of experimental outcomes:

- $\{|000\rangle\,,\ |111\rangle\}:\ \frac{9}{16}$
- $\{|001\rangle\,,\ |110\rangle\}:\ \frac{3}{16}$
- $\{|011\rangle\,,\ |100\rangle\}:\ \frac{3}{16}$
- $\{|010\rangle\,,\ |101\rangle\}:\ \frac{1}{16}$

This gives the wavefunction

$$
\begin{aligned}
|\Psi\rangle = \sqrt{\frac{9}{32}}\,(|000\rangle + |111\rangle) + \sqrt{\frac{3}{32}}\,(|001\rangle + |110\rangle) + \\
\sqrt{\frac{3}{32}}\,(|011\rangle + |100\rangle) + \sqrt{\frac{1}{32}}\,(|010\rangle + |101\rangle)
\end{aligned}
\tag{1}
$$

We know from the Malus law, that the weight of the case $a = c$ is $\frac{4}{16} = \frac{1}{4} = \cos^2 \angle 60$. But if I sum up the probabilities/weights of the case $a = c$, it yields $\frac{9}{16} + \frac{1}{16} = \frac{10}{16}$ and this is inevitable, what implies the impossibility of three entangled photons.

So Bell did not proof the impossibility of the photon as further defined by a hidden parameter λ, denoted as (λ), but from his thoughts about the 0°-30°-60°-game lead us to the impossibility of a tripel $\{(\lambda)\,,\ (\lambda)\,,\ (\lambda)\}$.

2 The general Situation: ϕ_1, ϕ_2 and ϕ_3 instead of 0°, 30° and 60°

- ϕ_1 channel: $\text{photon}_1 = (\lambda) \to \phi_1$-polarizer
- ϕ_2 channel: $\text{photon}_2 = (\lambda) \to \phi_2$-polarizer
- ϕ_3 channel: $\text{photon}_3 = (\lambda) \to \phi_3$-polarizer

Analogue to the above settings:

- $(a = b) \wedge (b = c) \implies (a = c)\,,\ \cos^2\phi_1 \cdot \cos^2\phi_2$
- $(a = b) \wedge (b \neq c) \implies (a \neq c)\,,\ \cos^2\phi_1 \cdot \sin^2\phi_2$
- $(a \neq b) \wedge (b = c) \implies (a \neq c)\,,\ \sin^2\phi_1 \cdot \cos^2\phi_2$
- $(a \neq b) \wedge (b \neq c) \implies (a = c)\,,\ \sin^2\phi_1 \cdot \sin^2\phi_2$

Furthermore we know by Malus with respect to the results a nd c:

- $(a = c)\,,\ \cos^2\phi_1 \cdot \cos^2\phi_2 + \sin^2\phi_1 \cdot \sin^2\phi_2 = \cos^2\phi_3$
- $(a \neq c)\,,\ \cos^2\phi_1 \cdot \sin^2\phi_2 + \sin^2\phi_1 \cdot \cos^2\phi_2 = \sin^2\phi_3$

Now this is very interesting. Of course it has to be $\phi_3 = \phi_1 + \phi_2$. So we have

- $\cos\phi_3 = \cos\phi_1 \cdot \cos\phi_2 - \sin\phi_1 \cdot \sin\phi_2$
- $\sin\phi_3 = \sin\phi_1 \cdot \cos\phi_2 + \cos\phi_1 \cdot \sin\phi_2$

or

- $\cos^2\phi_3 = \cos^2\phi_1 \cos^2\phi_2 - 2\cos\phi_1 \cos\phi_2 \sin\phi_1 \sin\phi_2 + \sin^2\phi_1 \sin^2\phi_2$
- $\sin^2\phi_3 = \sin^2\phi_1 \cos^2\phi_2 + 2\sin\phi_1 \cos\phi_2 \cos\phi_1 \sin\phi_2 + \cos^2\phi_1 \sin^2\phi_2$

It is clear, that in general the terms $2\cos\phi_1 \cos\phi_2 \sin\phi_1 \sin\phi_2$, $2\sin\phi_1 \cos\phi_2 \cos\phi_1 \sin\phi_2$ are not zero, so in general the identities above are not fulfilled. But for the cases

- $\phi_1 = 0$
- $\phi_1 = \frac{\pi}{2}$
- $\phi_2 = 0$
- $\phi_2 = \frac{\pi}{2}$

the identities are fulfilled. In the first case - i.e. $\phi_1 = 0$ - for example we have

- 0° channel: $\text{photon}_1 = (\lambda) \rightarrow$ 0°-polarizer
- 0° channel: $\text{photon}_2 = (\lambda) \rightarrow$ 0°-polarizer
- ϕ channel: $\text{photon}_3 = (\lambda) \rightarrow \phi$-polarizer

with the probabilities

- $(a = b)$: 1
- $(b = c)$: $\cos^2\phi$

- $(a = c): \ \cos^2 \phi$

and the wavefunction

$$|\Psi\rangle = \frac{\cos\phi}{\sqrt{2}} \left(|000\rangle + |111\rangle\right) + \frac{\sin\phi}{\sqrt{2}} \left(|001\rangle + |110\rangle\right) \tag{2}$$

There seems to be a kind of pauli law, which prohibits in general tripels of entangled photons, but for special polarizer settings there could be a possibility for the existence of such tripels. So if one starts with the settings $(0,\ 0,\ \phi)$ and then changes one of the both zero angels, it will end up the state of entanglement of the three photons. Very weird, isn't it?