On the Versatile Effects of Common Market Conditions on Market Inequality

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Abstract

Understanding the driving forces of market inequality is a core task. This paper seeks to study if and how changing market conditions influence equilibrium market inequality between ex ante heterogeneous agents that are engaged in some form of competition. By representing the model as a *competition for market shares* we derive a set of results that yield a tractable inequality analysis and help to identify structural connections between different competition models that allow for a unified treatment. We apply our results to competition theory, trade, consumption and income inequality, political economics and marketing, and connect some of the predicted inequality effects to empirical evidence.

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1 Introduction

Empirical studies commonly document that key quantities like market shares, payoffs, revenues, consumption or income feature substantial inequality in their dispersions. On the one hand, such inequality reflects ex ante differences in endowments, abilities, information, inclinations or alike of the respective agents. On the other hand, inequality may depend on market conditions that affect all agents. In case of firms such conditions could amount to consumer income, the industry level of productive efficiency, sales taxes, the amount of available production resources, the intensity of preferences, or advertising affinity.

In this article, we study how market inequality depends on conditions that are common to all agents. When can changes in such market conditions be exploited by some agents to increase, e.g., their market shares, and what type of changes induce more market equality via competitive forces? Can common inequality patterns be identified across different competition models? How are changes in the dispersion of market shares related to those of payoffs, revenues or expenditures?

We study such questions by equivalently representing a competitive situation as a *competition for market shares*, where market shares refer to the dispersion of quantities such as sales or consumption. In this representation, the agents solve their respective optimization problems by directly choosing their market shares, rather than indirectly via their choices of "actions". This transformation of the relevant optimization problems allows us to derive intuitive and application-friendly results from the equilibrium conditions to systematically study how market inequality evolves if market conditions change. We show that our framework integrates prominent models, such as monopolistic competition, perfect competition, general equilibrium, or competition for prizes as special cases. It thereby allows us to identify structural relations between different forms of competition, which permit a unified and tractable inequality analysis.

We say that a market condition x – essentially an exogenous parameter that enters the payoff functions of some or all agents – induces inequality effects whenever the equilibrium market shares are not invariant to x. Our first results establish that a simple property – the *direct-aggregative effect* – of a single representative equilibrium equation, which originates from agent-wise optimality and is recursive in the agents' types, is necessary and sufficient for the existence of inequality effects. This real number basically summarizes the differences of how a change in x affects marginal costs and benefits of two agents directly and via the market response of all agents. For example, a positive direct-aggregative effect means that one agent is more positively (or less negatively) affected by a change of x than the other, ceteris paribus. Condition x induces inequality effects *iff* the direct-aggregative effect is non-zero.

The direct-aggregative effect also is informative about qualitative properties of the inequality effects that can arise. Of particular interest is the case where the direct-aggregative effect has the same sign for all agents. This typically occurs with changes in *common market conditions*, which are those that enter the payoff functions of each agent. For example, an increase in disposable consumer income affects all sellers in a given market, meaning that each of them strives for a larger market share. We would like to know: does this inclination play out to reinforce or alleviate market inequality in equilibrium? In our second set of results, we prove that whenever the direct-aggregative effect is strictly positive (negative) for all agents, the inequality effects must induce a clockwise (counter-clockwise) *rotation* of the equilibrium market shares, meaning that market inequality must increase (decrease) according to every Lorenz-consistent inequality measure. This result is of practical relevance because the direct-aggregative effect indeed is sign-uniform in many important cases.

Our inequality analysis further identifies a significant role of power functions. If the type-recursive equation that characterizes the direct-aggregative effect is such that the "costs" and "benefits" associated with attaining a certain market share are power functions, and the direct-aggregative effect is sign-uniform for all agents, then any inequality effect must take on the form of a rotation that is *monotonic in agent types*. That is, the changes in market shares are ordered such that stronger agents must gain (or lose) more, in the relative sense, compared to weaker agents.

Power functions also allow to say more, e.g., about the relation between market and payoff shares. For example, we show that with equal revenues per unit of market shares, competition must play out such that market and payoff shares coincide. By contrast, if those agents with larger market shares also earn higher revenues per unit of market share, payoff shares must be less equally dispersed than market shares. The results on power function are of interest for applied work, as power functions arise in many economic models (Newman, 2005).

Our approach equips us with powerful tools for studying equilibrium inequality, because the direct-aggregative effect is a local property, determined by a single equation, which nevertheless is informative about global properties of a distribution. To obtain more specific insights about market inequality, and to vindicate the applicability of our approach, we study several applications from different fields. We thereby distinguish whether payoff functions feature *common or idiosyncratic valuations* per unit of market share. The former represents a situation where heterogeneous agents face a *symmetric* type of competition. Monopolistic competition, perfect competition or homogeneous-valued contests are examples of this setting, and allow for a unified inequality analysis.

This analysis reveals that the inequality effects induced by common market conditions depend on whether there is ex ante agent heterogeneity in the returns to scale associated with the "production" of market shares. For example, we establish that common demand or cost "shifters" cannot induce inequality effects in the knife-edge case where all agents are subject to perfectly identical scale effects. By contrast, such shifters are likely to trigger a rotation of market shares if the agents differ in their scale effects. As an application, this result complements demand-side explanations, such as Mrázová and Neary (2017), for why an increasing international integration may have fostered firm-side market inequality.

The above insights extend to idiosyncratic valuations if, for all agents, these valuations respond by *equal proportions* to changes in x. By contrast, if valuations change by different proportions, additional inequality effects emerge. For example, if valuations increase by larger proportions for agents with larger (smaller) market shares, then a clockwise (counter-clockwise) rotation occurs under the conditions that were to preserve equilibrium inequality with common valuations.

These results are helpful to study, e.g., the inequality effects that arise *within* the firm- and consumer-side in general equilibrium models. The reformulation of the model as competition for market shares reveals that consumer-side income or consumption inequality effects are equivalent to those in contests with idiosyncratic valuation functions. Moreover, these effects are sensitive to whether the ex ante consumer heterogeneity originates from resource endowments, such as units of effective labor, or from capital income. We use this central observation to analyze the connections between consumption and leisure inequality (Attanasio and Pistaferri, 2016), or to study consumer-side inequality effects, e.g., of an increasing productive efficiency, or of an "unconditional basic income", financed by a market-based tax, as discussed in several European countries.

From an overarching perspective, our approach differs from, but also complements, a recent literature on equilibrium inequality that focuses on how the dispersion of certain parameters constituting the *ex ante agent heterogeneity* map into the dispersion of certain equilibrium quantities for a fixed market structure. For example, Mrázová et al. (2016) relate the distribution of firm sales and markups to the underlying distribution of technology in a monopolistic competition setting, or Jensen (2017) studies how exogenous changes in the ex ante distribution, such as increased uncertainty in a prior belief, may affect certain outcomes. By contrast, our results are about when changes in market conditions make (dis-)advantaged agents more or less dominant, leading to more or less market concentration, for given ex ante agent heterogeneity. We apply our insights to various central applications, and connect some of the more specific results with the relevant literature. While our main focus is on the inequality effects induced by common market conditions, we emphasize that our approach is also suitable to study the inequality effects induced by certain changes in the ex ante agent heterogeneity itself; see Section 5.3. We also point out that our approach does not rely on monotonic relations between exogenous parameters and equilibrium actions, which is the central tenet of the literature on monotone comparative statics. Importantly, the mere fact that, e.g., equilibrium actions or payoff functions could be monotonic in x for all agents does not generally pin down the inequality effects that can arise.

Finally, our approach has the merit of providing a simple, powerful alternative for studying inequality effects. In procedural terms, our inequality analysis can be summarized as: 1) restate the optimization problem (e.g., the payoff function) with the agent's market share, $p(i)$, as the choice variable; 2) derive the first-order optimality conditions wrt $p(i)$; 3) use our formal tools to identify from the optimality equation whether a change in x induces inequality effects, and if these inequality effects take on the form of rotations of the market shares.

We define the concept of a competition for market shares in Section 2, where we also introduce our notion of ex ante agent heterogeneity ("agent types") and other key definitions. In

Section 3 we derive the main formal tools for our inequality analysis, and use those to obtain a number of general inequality predictions in Section 4. Finally, Section 5 evaluates our results in the context of specific applications. Proofs are in Appendix A, while Appendix B contains additional results complementing our general analysis.

2 Competition for Market Shares

Let $I = [0,1]$ denote a set of agents For each agent $i \in I$, the payoff function is $\Pi(i) = p(i)V(i) - \Phi(i)$, where $p(i)$ is the market share of i, $V(i)$ the value earned per unit of market share, and $\Phi(i)$ are expenditures. Each agent can choose an action variable $t(i) \geq 0$ to maximize $\Pi(i)$. In general, $t(i)$ can enter each component of $\Pi(i)$, together with the actions of all agents summarized by a quantity $T \in \mathbb{R}_+$. The payoff function of at least one agent depends on an exogenous parameter $x \in X$. The set X is an open interval in $\mathbb{R}$, and captures a market condition of interest. Thus, the payoff function generally is of the form

$$\Pi(i, t(i), T; x) = p(i, t(i), T; x)V(i, t(i), T; x) - \Phi(i, t(i); x). \tag{1}$$

The way how $t(i)$ or T enter $\Pi(i)$ pins down the details of the competition faced by the agents. In several of our applications, T is the sum of all chosen actions. That is, $T = \int_I t(i)di$, where the actions $t(i)$ chosen by each agent are summarized by an action profile $t : I \to \mathbb{R}_+$. More generally, T is determined by an aggregator function $Z(t) = T$ for a set of viable action profiles.[1] As $p(i)$ are market shares, we impose the identifying assumption that they must add up to one

$$T = Z(t) \iff \int_I p(i, t(i), T; x)di = 1. \tag{2}$$

Condition (2) states that all market shares must sum to one for given T iff the action profile t generates T. We call $(\{\Pi(i)\}_{i\in I}, Z)$ a *competition for market shares* if each $\Pi(i)$ is of form (1), and $Z(\cdot)$ verifies (2). Many conventional models are competitions for market shares; see Section 5. We further note that our notion of a competition for market shares can easily be adopted to constrained optimization problems, such as conventional consumer utility maximization (see Section 5).

2.1 Equilibrium Market Shares

Given a competition for market shares $(\{\Pi(i)\}, Z)$, an *equilibrium* is an action profile t and a quantity T, where $t(i)$ maximizes (1) given T $\forall i \in I$ and T is endogenously determined by $Z(t) = T$. This equilibrium definition encompasses, e.g., monopolistic competition equilibria, Walrasian equilibria, or equilibria in large aggregative games (see Section 5). Further, it is consistent with equilibria in the "Global Games" literature (see, e.g., Morris and Shin, 2002). The common aspect of all these equilibrium notions is that the agents take the aggregate T

[1] Let $\mathbf{T}$ be the set of all action profiles $t : I \to \mathbb{R}_+$. A given aggregator function Z is a mapping $Z : \mathbf{T}_Z \to \mathbb{R}_+$ defined on a subset $\mathbf{T}_Z \subset \mathbf{T}$. An action profile is ($Z$-)viable if $t \in \mathbf{T}_Z$. For example, if $Z(t) \equiv \int_I t(i)di$, then $\mathbf{T}_Z$ consists of all (Riemann)-integrable functions $t : I \to \mathbb{R}_+$.

as given when maximizing, while T is endogenous to the model.[2] In Appendix B.5 we show that our approach is not confined to this assumption, as it can also encompass the case of Nash equilibria in aggregative games.

Competition for Market Shares An equilibrium (t, T) comes with a certain dispersion of market shares $p(\cdot)$. To systematically study how a market condition x may affect the prevailing market inequality, we pursue an equivalent characterization of equilibrium, where the agents compete directly in market shares, rather than indirectly by choosing them via $t(i)$. It turns out that this alternative equilibrium characterization delivers a tractable, powerful way of analyzing how x affects equilibrium inequality.

Let the market share function $p(i, t(i), T; x)$ in (1) be $t(i)$-bijective for each $T > 0$, $x \in X$ and $i \in I$. Thus, for given (i, T, x), any choice of action $t(i) \in \mathbb{R}_+$ has a unique number $p(i) \in \mathbb{R}_+$ associated with it. By change-of-variable, we may then rewrite payoff (1) as

$$\Pi(i, p(i), T; x) = p(i)V(i, p(i), T; x) - \Phi(i, p(i), T; x) \tag{3}$$

with some (convenient) abuse of notation.[3] To further ease notation, we often abbreviate payoff function (3) as $\Pi(i)$. We now define an equilibrium as a situation, where $p(i) > 0$ maximizes (3) for each agent i, and market shares integrate to one.

Definition 1 (Equilibrium) *An equilibrium is a pair $(p(\cdot), T)$ consisting of a market share function $p : [0, 1] \to \mathbb{R}_{++}$ and a number $T \in (0, \infty)$ such that*

i) For each $i \in I$, $p(i)$ solves $\max_{p(i)\geq 0} \Pi(i)$, where $\Pi(i)$ is given by (3).

ii) $\int_I p(i)di = 1$

Whenever $(p(i), T)$ is an equilibrium in Definition 1, the unique actions $t(i)$ corresponding to the equilibrium market shares $p(i)$ together with $T = Z(t)$ must be an equilibrium in the original sense, and vice-versa.[4] We restrict attention to interior equilibria, where each agent obtains a positive equilibrium market share.[5]

What makes the transformed version most convenient for our inequality analysis is that it leads to a simple, type-recursive equilibrium equation identifying the inequality effects caused by x, without the need to "solve" the model for $t(i)$ or T. This equation produces a

[2]Such a property appears particularly reasonable with a large number of agents (Alos-Ferrer and Ania, 2005; Acemoglu and Jensen, 2010; Hefti, 2016; Camacho et al., 2018), and it simplifies the formal analysis of the inequality effects that this paper cares about.

[3]If $p = p(i, t, T)$ then $t \equiv p^{-1}(i, p, T)$ is the inverse of the function $p(i, t, T)$ with respect to the variable t. Hence $\Pi(i, t(i), T) = \Pi\left(i, p^{-1}(i, p(i), T), T\right) \equiv \hat{\Pi}(i, p(i), T)$. The abuse of notation is that we continue to use the notation $\Pi(i, p(i), T)$ (instead of $\hat{\Pi}(\cdot)$) in the transformed problem (similarly, we use V and Φ instead of $\hat{V}$, $\hat{\Phi}$).

[4]This is a consequence of the bijective relation between actions and market shares, which we formally prove in Appendix B.1.

[5]This is not critical as we consider a fixed set of agents; see Hefti and Lareida (2020) for an application with agent entry. It is also possible to extend our analysis to the case where the market shares of some agents become zero – the economics leading to such a situation are similar to those that generally imply an increasing inequality.

set of simple but powerful analytical results that can be directly applied to study inequality, which substantially increases the tractability of the analysis.

Continuum Agents A final remark concerns the use of "continuum" agents. We show in Appendix B.3 that continuum agents are without loss of generality in such that, for a given number of atomistic agents $n \in \mathbb{N}$, the corresponding "discrete" market shares $p^d(i)$ can be identified from the respective equilibrium step-density function $p(i)$ by rescaling the latter.[6] Intuitively, these steps represent the different agent types as specified by the ex ante agent heterogeneity (see Section 2.3). For a finite number of different agent types our inequality results can be adjusted to the atomistic case simply by means of the natural modifications.[7] The continuum approach has the formal advantage that the equilibrium $p(\cdot)$ is a (Lebesgue) density $p : [0,1] \to \mathbb{R}_+$ with a *fixed support* for any number of agent types, rather than a discrete mapping with variable support depending on the number of agents. This simplifies exposition, and also includes the case of "true continuum agents", where $p(\cdot)$ is a continuous.

2.2 Regularity Conditions

By Definition 1, the equilibrium market share $p(i)$ maximizes $\Pi(i)$ at the equilibrium quantity T for each agent i. To study this maximization problem with calculus, we set

$$g(i,p,T;x) \equiv \frac{\partial\left(pV(i,p,T;x)\right)}{\partial p}, \qquad \varphi(i,p,T;x) \equiv \frac{\partial \Phi(i,p,T;x)}{\partial p},$$

where $g(\cdot)$ can be interpreted as marginal benefits, and $\varphi(\cdot)$ as marginal costs, respectively, pertaining to an aspired market share p. For given i and T, the optimality conditions at an interior maximizer $p(i) > 0$ then simply are

$$g(i,p(i),T;x) = \varphi(i,p(i),T;x), \tag{4}$$

or, in short-hand, $g(i) = \varphi(i)$. To assure existence of a unique equilibrium $(p(\cdot),T)$, and to keep complexity of the inequality analysis at a minimum, we impose conventional regularity conditions.

Assumption 1 (Regularity) *For each $i \in [0,1]$ and any $p,T > 0$, $g(\cdot)$ and $\varphi(\cdot)$ in (4) are C^1-functions of x. Further, the following properties hold $\forall x \in X$ and each $i \in [0,1]$:*

(A1) $\forall T > 0$ and $\forall p \geq 0$: $\Pi(i)$ in (3) is a strongly quasiconcave C^2-function in p, and

- *$g(i,0,T;x) > 0$ and $g(i,\cdot,T;x)$ are bounded from above*
- *$\varphi(i,0,T;x) = \Phi(i,0,T;x) = 0$, $\varphi_p(i,p,T;x) > 0$, and $\lim_{p\to\infty} \varphi(i,p,T;x) = \infty$*

(A2) $\forall p > 0$: $g(i,p,\cdot;x)$ and $\varphi(i,p,\cdot;x)$ are C^1-functions of T, and

[6] To illustrate, if $n = 3$ and $p^d(1) = 1/2$, $p^d(2) = 1/3$, $p^d(3) = 1/6$, then $p(i) = 3/2$, $i \in [0,1/3)$, $p(i) = 1$, $i \in [1/3,2/3)$ and $p(i) = 1/2$, $i \in [2/3,1]$, and $\int p(i)di = 1/3(3/2+1+1/2) = 1$.

[7] For example, in the atomistic case we would replace condition $\int p(i)di = 1$ in Definition 1 by $\sum_{i=1}^n p(i)$, where $p(i)$ is a density with respect to the counting measure.

- *$g(i,1,0;x) > 0$ and $g(i,1,\cdot;x)$ are bounded from above*
- *$\varphi(i,p,0;x) = 0$, $\varphi_T(i,p,T;x) > 0$ and $\lim_{T\to\infty} \varphi(i,p,T;x) = \infty$*
- *$g_T(i,p,T;x) < \varphi_T(i,p,T;x)$ whenever $g(i,p,T;x) = \varphi(i,p,T;x)$*

Figure 1 illustrates some key implications of these assumptions. In essence, (A1)-(A2) are slope and "Inada"-conditions assuring that the functions $g(\cdot)$ and $\varphi(\cdot)$ are well-behaved. (A1) implies that, for given $T > 0$, a single crossing of $g(\cdot)$ with $\varphi(\cdot)$ exists (Figure 1: Left), which pinpoints the optimal solution $p(i;T)$ to (4). Assumption (A2) further implies the existence of a unique $T > 0$ such that $\int p(i;T)di = 1$, which pins down the equilibrium T (Figure 1: Left).[8]

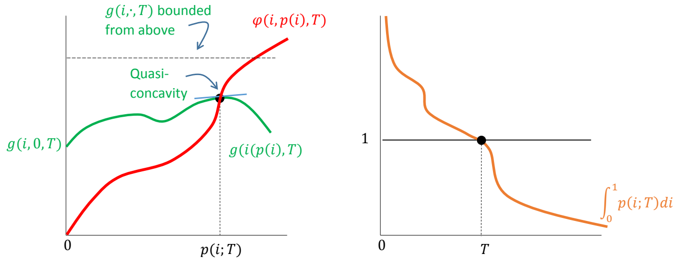

Figure 1: Illustration of Assumption 1

2.3 Ex Ante Agent Heterogeneity

We suppose that the agents differ ex ante, e.g., in their cost functions, production possibilities or alike. Formally, such disparities are manifested through differences in (marginal) cost or benefit functions, which we make precise next.

Assumption 2 (Ex ante Heterogeneity) *$\forall p, T > 0$ and $x \in X$: $pV(i,p,T;x)$ and $g(i,p,T;x)$ are (weakly) decreasing, and $\Phi(i,p,T;x)$ and $\varphi(i,p,T;x)$ (weakly) increasing in i.*

This assumption implies that agents are sorted left-to-right, where i never features lower (marginal) benefits and never higher (marginal) costs than j for $j > i$. An example is that agents differ only in their ability to compete for an object of common value, such that

$$\Pi(i) = p(i)V(p(i),T) - c(i)\Phi(p(i),T), \tag{5}$$

where $c(i)$ is (weakly) increasing in i. The main implication of Assumption 2 is that ex ante heterogeneity leads to similarly ordered ex post heterogeneity, where agents with a lower index i achieve (weakly) larger equilibrium market shares and payoffs.[9]

[8]See Appendix B.2 for a proof of equilibrium existence and uniqueness. It is known that equilibria can exist under weaker conditions than imposed here. Nevertheless, we regard Assumption 1 as a natural starting point as analyzing equilibrium inequality in itself is sufficiently complex.

[9]Assumption 2 actually is stronger than what our inequality results require. We shall only need that the equilibrium order of the market shares is not pivoted by the parameter x, i.e., if $p(i;x) \geq p(j;x)$ then also $p(i;x') \geq p(j;x')$ $\forall x' \in X$ and any i,j.

Proposition 1 (Ex post Heterogeneity) *Let $(p(\cdot), T)$ be an equilibrium, and consider any two agents with $j > i$. Then under Assumptions 1 and 2:*

i) ***No "leap-frogging"**: $p(i) \geq p(j)$ and $\Pi(i) \geq \Pi(j)$.*

ii) ***Strict order**: $p(i) > p(j)$ if $\forall p, T > 0$ we have $g(i, p, T) \geq g(j, p, T)$ and $\varphi(i, p, T) \leq \varphi(j, p, T)$ with at least one inequality strict; and $\Pi(i) > \Pi(j)$ if $\forall p, T > 0$ we have $pV(i, p, T) \geq pV(j, p, T)$ and $\Phi(i, p, T) \leq \Phi(j, p, T)$ with at least one inequality strict.*

iii) ***Equality**: $p(i) = p(j)$ if both $g(i, p, T) = g(j, p, T)$ and $\varphi(i, p, T) = \varphi(j, p, T)$ $\forall p, T > 0$.*

The "sorting" property i) follows from individual optimality: If, by contradiction, $p(i) < p(j)$, then the ex ante stronger agent i could always benefit from deviating to $p(j)$, hence there cannot be such deviations in equilibrium; similar reasons apply to the remaining claims.

In view of iii), we classify all agents $i \in [0, 1]$ with identical payoff functions (3) as being of the same (ex ante) *type*.

Definition 2 (Agent Type) *Two agents i, j are of the same type if $\Pi(i, p, T; x) = \Pi(j, p, T; x)$ $\forall p, T, x$.*

By Proposition 1, the equilibrium market share function $p(\cdot)$ is sorted according to the different agent types. Formally, for a given agent $i \in [0, 1]$, the subset $[i] \equiv \{s \in [0, 1] : p(s) = p(i)\}$ is the equivalence class of agents with representative i. Thus, all agents in a given equivalence class $[i]$ are of the same type, and the collection of all equivalence classes corresponds to the set of agent types. Based on this insight, it makes sense to define a transitive relation $\rhd$ by

$$j \rhd i :\Longleftrightarrow j > i \text{ and } j \notin [i].$$

The relation $\rhd$ allows for an easy distinction between different agent types. We distinguish the following two central cases:

Definition 3 *Let $p : [0, 1] \to \mathbb{R}_{++}$ be a density function. Then*

*I) $p(\cdot)$ belongs to **Class I** if $p(\cdot)$ is step-wise decreasing, right-continuous with at least one and at most finitely many downward steps.*

*II) $p(\cdot)$ belongs to **Class II** if $p(i)$ is a C^1-function with $p'(i) < 0$ for $i \in [0, 1]$.*

To help understand these notions, observe that if there are *finitely many different agent types* this implies, by Proposition 1, that the equilibrium market share function $p(\cdot)$ must be a *decreasing step-density*, i.e., a member of Class I. The steps of $p(\cdot)$ then exactly correspond to the different agent types, and $j \rhd i$ means that j is of a "weaker" type than i, i.e., j sits on a lower step than i. This matters because, as mentioned earlier, it is the continuum analogue to the case of "atomistic" agents.

As an example, suppose that agent payoffs are given by (5), where the agents differ in their cost coefficients $c(i)$. If there are $n \in \mathbb{N}$ different cost types, this means that $c(i)$ is an increasing step function with n steps, and $p(\cdot)$ must be a Class I density with n steps; see

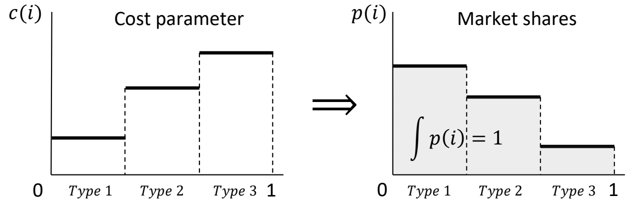

Figure 2: Three different agent types ($n = 3$)

Figure 2. By contrast, with *true continuum agents*, $c(i)$ is strictly increasing, such that $p(\cdot)$ must be a Class II density, in which case the relations "$\rhd$" and "$>$" coincide.[10]

We finally remark that if $p(\cdot)$ belongs either to Class I or II, then $p(\cdot)$ must display the "somewhere strictly decreasing" (SSD) property: $\exists i_0 \in (0,1)$: $p(i) > p(j)$ for $i < i_0 \leq j$. That is, $p(\cdot)$ cannot be constant – we exclude the trivial case where market shares are uniformly dispersed because that all agents are one and the same type.

3 Inequality Effects

In this section we derive our main inequality tools. While these tools, once available, are readily applicable, proving their validity in a general context is not trivial, and this section wants to pay justice to the latter. Specifically, Section 3.1 summarizes our main definitions regarding inequality effects, and Sections 3.2 - 3.3 develop the key results about the existence of inequality effects and rotations. Section 3.4 presents a simple procedure for determining $T'(x)$, and Section 3.5 provides concluding remarks.

A reader mostly interested in the applications of our tools may want to get familiar with the definitions in Section 3, and then move to Section 4 which demonstrates how to apply our formal results in a general economic context.

3.1 Main Defintions

The market condition x *induces inequality effects* if the equilibrium market share function $p(\cdot)$ is not constant in x.

Definition 4 (Inequality Effects) *Let* $(\{\Pi(i)\}, Z)$ *be a competition for market shares. The parameter* x ***induces inequality effects*** *if* $\exists x, x' \in X$: $p(\cdot; x) \neq p(\cdot; x')$. *If* $p(\cdot; x) = p(\cdot; x')$ $\forall x, x' \in X$, *then* x *is* ***inequality preserving***.

We are particularly interested in those inequality effects that do not permute how the agents are sorted in equilibrium if x changes. In view of the common market conditions – those that enter the payoff functions of all agents – we aim to study, this is an intuitive refinement, and one that naturally holds with the conditions of interest in our later applications. To make these notions precise, let $p(\cdot; x)$ denote the equilibrium market share function for given $x \in X$. For a given i, recall that $[i]_x \equiv \{s \in [0,1] : p(s) = p(i)\}$ is the subset of agents

[10] In the working paper version, we show that the CDF associated with $p(\cdot)$ must be increasing, concave and strictly above the diagonal for $i \in (0,1)$ if p is of Class I or II.

that are of type i, given $x \in X$. Thus, $\mathcal{T}_x \equiv \{[i]_x : i \in [0,1]\}$ is the equilibrium set of agent types given $x \in X$, and the map $x \mapsto \mathcal{T}_x$ is well-defined as, under Assumptions 1-2, a unique equilibrium market share function $p(\cdot; x)$ exists for each $x \in X$. The notion of an *order-preserving market condition* means that this assignment is constant.

Definition 5 (Order-Preserving Market Condition) *x is an order-preserving market condition if $\mathcal{T}_x = \mathcal{T}_{x'}$, $\forall x, x' \in X$.*

In words, whenever an agent i obtains a larger equilibrium market share than j for some x, then this must hold for any $x \in X$ if x is an order-preserving market condition. This does not exclude, however, that x might affect only a single agent's payoff, i.e., is an idiosyncratic rather than a common condition. We only require that changes in x cannot induce equilibrium leap-frogging by some agents.[11]

Definition 6 (Common Market Condition) *An order-preserving market condition x is a common market condition if for any given $p, T > 0$*

- *$g(i, p, T; x)$ either increases, decreases or remains constant in x $\forall i \in [0,1]$, and*
- *$\varphi(i, p, T; x)$ either increases, decreases or remains constant in x $\forall i \in [0,1]$.*

The decisive aspect of a common market condition is that it affects marginal costs and benefits of each agent in a similar way. Common market conditions are interesting because if they change, all agents strive to adjust their market shares in the same direction, making the equilibrium implications for market inequality non-trivial. In our later applications, an increase in the industry-level productive efficiency, or a change in a sales tax are examples for common market conditions.

3.2 Existence of Inequality Effects

Our first result characterizes when x induces inequality effects. Throughout this section, we take Assumptions 1-2 as satisfied.

Let $x_0 \in X$. As (4) holds $\forall i$ in equilibrium, it follows that equilibrium forces equate the ratio of marginal benefit over marginal costs for any two agents. Hence

$$\frac{g(i, p(i), T; x_0)}{\varphi(i, p(i), T); x_0)} = \frac{g(j, p(j), T; x_0)}{\varphi(j, p(j), T); x_0)} \qquad i, j \in [0,1], \tag{6}$$

or, in short-hand, $g(i)/\varphi(i) = g(j)/\varphi(j)$. We exploit (6) to obtain a single equilibrium equation characterizing the inequality effects induced by x. Define

$$dp(i) \equiv \frac{\partial p(i; x_0)}{\partial x}, \qquad \Delta_i \equiv \frac{dp(i)}{p(i)}, \qquad \varepsilon_i \equiv \frac{g_p(i)p(i)}{g(i)}, \qquad \eta_i \equiv \frac{\varphi_p(i)p(i)}{\varphi(i)}. \tag{7}$$

Thus, Δ_i is the percentage change in i's market share, while ε_i, η_i are the p-elasticities of marginal benefits $g(\cdot)$ and costs $\varphi(\cdot)$, respectively. The following result identifies a simple affine relation between Δ_i, Δ_j of any two different agent types i, j.

[11] Note that this can always be assured, e.g., if $p(\cdot)$ is Class I and the changes in x remain sufficiently small.

Lemma 1 *Let $j \triangleright i$, and consider a marginal change $dx > 0$. Then $\eta_i > 0$, $\eta_i > \varepsilon_i$, and*

$$\Delta_i = \Delta_j k_{ij} + R_{ij}, \qquad k_{ij} \equiv \frac{\eta_j - \varepsilon_j}{\eta_i - \varepsilon_i} > 0, \tag{8}$$

$$R_{ij} \equiv \frac{A(i) - A(j)}{\eta_i - \varepsilon_i}, \qquad A(s) \equiv \left(\frac{g_T(s)}{g(s)} - \frac{\varphi_T(s)}{\varphi(s)} \right) T'(x_0) + \left(\frac{g_x(s)}{g(s)} - \frac{\varphi_x(s)}{\varphi(s)} \right). \tag{9}$$

The type-recursive equation (8) decomposes the relation between Δ_i and Δ_j into a *direct-aggregative effect* (R_{ij}), and an *indirect effect* (k_{ij}). To understand these names, note from (9) that R_{ij} depends on x directly via $g_x(s)$ and via the aggregate quantity $T(x)$. By contrast, k_{ij} collects the indirect effects that dx has on $g(\cdot)$ and $\varphi(\cdot)$ via the changes in $p(i)$. We shall simply write R and k if there is no confusion about types. Decomposition (8) is key for analyzing if and how x affects $p(i)$. The first theorem shows that inequality effects exist *iff* R_{ij} is non-zero for at least two different agent types.

Theorem 1 (Existence of Inequality Effects) *If $R_{ij} = 0$ $\forall i, j \in [0, 1]$ and any $x \in X$, then x is inequality preserving. Conversely, if for a given $x \in X$ $\exists i, j \in [0, 1]$ such that $R_{ij} \neq 0$, then inequality effects arise. Specifically, $\exists \delta > 0$ such that $p(\cdot; x') \neq p(\cdot; x)$ for any $x' \in (x - \delta, x + \delta) \backslash \{x\}$.*

We note from Theorem 1 that the indirect effect k_{ij} plays no role for whether inequality effects arise. This effect only captures how sensitive $g(\cdot), \varphi(\cdot)$ respond to changes in p, and as such can influence certain quantitative aspects should inequality effects occur (see below), but not whether they occur in the first place.

3.3 Rotations

If x induces inequality effects, what more can be said about their properties? Because the direct-aggregative effect R_{ij} is decisive for whether inequality effects arise, it is a natural starting point to study those inequality effects more carefully that arise if R_{ij} is either positive or negative for all agents – a feature that will be relevant for common market conditions. Our main result below shows that, in these cases, the inequality effects must be described by *rotations* of $p(\cdot)$. We take Assumptions 1-2 as satisfied, and additionally let x be an order-preserving market condition as in Definition 4. Further, we take the ex ante agent heterogeneity to be such that $p(\cdot)$ is a Class I or II density in equilibrium.

Definition 7 (Rotations) *Let $x, x' \in X$ and $p(\cdot, x)$, $p(\cdot, x')$ be two decreasing densities with support $[0, 1]$. We say that $p(\cdot; x')$ is an **outward-rotation (OR)** of $p(\cdot; x)$, or $p(\cdot; x)$ is an **inward-rotation (IR)** of $p(\cdot; x')$, if $\exists$ $0 < i_0 \leq i_1 < 1$ such that*

$$\begin{aligned} p(i; x') &> p(i; x) \quad i \in [0, i_0) \\ p(i; x') &< p(i; x) \quad i \in (i_1, 1) \\ p(i; x') &= p(i; x) \quad i \in (i_0, i_1] \end{aligned} \tag{10}$$

where the last condition only applies if $i_0 < i_1$. We say that a parameter change $dx > 0$ induces an OR (IR) of $p(\cdot; x)$ if $\exists \delta > 0$ such that $p(\cdot; x')$ is OR (IR) of $p(\cdot; x)$ for any

$x' \in (x, x+\delta)$.

The defining property of a rotation is that there is a unique "turning point", where the market shares of all stronger agents increases (decreases), while it decreases (increases) for all weaker agents (see Figure 3). If $p(\cdot)$ rotates, this implies an increasing or decreasing

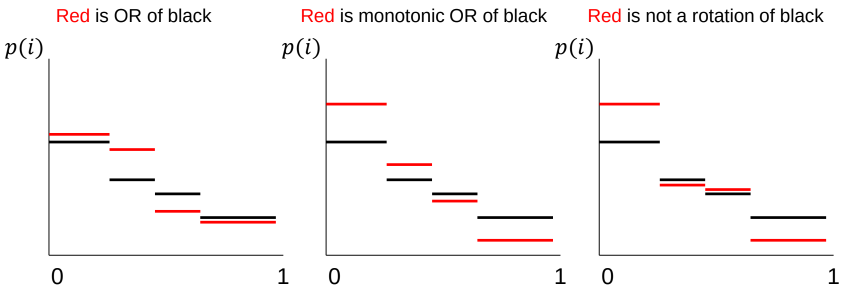

Figure 3: Rotations: Examples and counterexample

inequality in the corresponding distribution, depending on the type of rotation. Specifically, if $p(\cdot, x')$ is an OR (IR) of $p(\cdot, x)$, then the dispersion of market shares at x' is comparably less (more) equal, in thus that $p(\cdot; x')$ *Lorenz dominates* (is Lorenz dominated by) $p(\cdot, x)$.[12] Many standard inequality measures, such as the Gini coefficient or the notion of first-order stochastic dominance, are consistent with the partial order generated by Lorenz dominance (Atkinson, 1970). Thus, if $p(\cdot, x')$ is an OR (IR) of $p(\cdot, x)$, then $p(\cdot; x')$ is less (more) equally dispersed than $p(\cdot; x)$ according to any Lorenz-consistent inequality measure.

3.3.1 Existence of Rotations

We begin by establishing that the direct-aggregative effect R alone is decisive for whether a rotation arises.

Definition 8 *R is uniformly positive (negative) at $x_0 \in X$, if $R_{ij}(x_0) > (<) 0$ $\forall j \triangleright i$. Further, R is globally uniformly positive (negative) if $R_{ij}(x)$ is uniformly positive (negative) $\forall x \in X$.*

Theorem 2 (Rotational Effects) *If $p(\cdot)$ belongs to Class I and R is uniformly positive (negative) at $x_0 \in X$, then $dx > 0$ induces an OR (IR) of $p(\cdot, x_0)$. If $p(\cdot)$ belongs to Class I or II and R is globally uniformly positive (negative), then the market shares of the strongest types $i \in [0]$ increase (decrease) strictly in x, while the ones of the weakest types $i \in [1]$ strictly decrease (increase).*

To understand the first claim, let $j \triangleright i$ and note that $R_{ij} > 0$ iff $A(i) > A(j)$. As $A(i)$ captures how sensitively marginal costs and benefits respond to x and T, the last inequality intuitively states that the stronger agent type i has a stronger incentive to aspire for a larger market share (or a weaker incentive to reduce the market share). This means that if type i

[12] The Lorenz curve is a common tool in inequality analysis, and a distribution Lorenz dominates another distribution if its Lorenz curve lies below the Lorenz curve of the other. It is straightforward to verify that the Lorenz curve associated with $p(\cdot, x')$ is below the one of $p(\cdot, x)$ if the former is OR of the latter.

manages to increase its market share, this should hold for stronger type as well. Because not all market shares can increase, the resulting inequality effect takes on the form of a rotation.

The second result shows that the tails of the distribution evolve monotonically in x under the respective conditions. A corollary to this result is that the inequality effects in the two- or three-types cases are fully characterized.[13]

3.3.2 Monotonic Rotations

Rotations are a fairly general type of inequality effect that allows for a rich pattern of the "middling" agents. For example, an OR can be consistent with some winning agents "catching up" with even stronger types. This is illustrated in Figure 3 (left) where, among the winning agents, the second strongest agents gain more market share than the strongest, such that the gap between them narrows. By contrast, the gap between the agents on the winning and losing side, respectively, widens in the rotation of the middle panel of Figure 3. The following makes the notion of a rotation featuring such "increasing gaps" precise.

Definition 9 (Monotonic Rotations) *Suppose that $\infty > p(\cdot;x'), p(\cdot;x) > 0$ are right-continuous, decreasing SSD densities with the same equivalence classes $[i]$. If*

$$\frac{p(i;x')}{p(j;x')} > (<) \frac{p(i;x)}{p(j;x)} \quad \textit{whenever } j \rhd i \in (0,1) \tag{11}$$

*is satisfied, then $p(\cdot;x')$ is a **monotonic** OR (IR) of $p(\cdot;x)$.*

The fact that condition (11) indeed implies that a rotation must occur is proved in Appendix B.4.1.[14] An equivalent interpretation of (11) is that if $p(i;x')$ is a monotonic OR (IR) of $p(i;x)$, then the relative change in market shares is strictly increasing (decreasing) in agent type, such that the strongest agents ($i \in i[0]$) gain (lose) most while the weakest agents ($i \in i[1]$) lose (gain) most.[15]

If $p(\cdot;x')$ is a monotonic OR of $p(\cdot;x)$ and i,j are two different agents types both featuring higher market shares in the new equilibrium, then the absolute gap between these market shares must have widened. Thus, a "catching up" as in Figure 3 (left panel) is impossible. Formally, this follows from the following simple fact.

Lemma 2 *Consider real numbers with $u' > u > 0$ and $v' > v > 0$. If $\frac{u'}{v'} \geq \frac{u}{v} > 1$, then also $u' - v' > u - v$.*

[13]This follows because in the two-types case $R_{01} \geq (>) 0$ iff $R_{10} \leq (<) 0$, meaning that R is either uniformly positive (negative) or $R = 0$. Moreover, in the three-types case it follows from (the proof of) Theorem 2 that if R is globally uniformly positive (or negative), then any $x > x_0$ induces an OR (IR) of $p(\cdot, x_0)$, because the behavior of the "middle group" does not matter by Definition 7.

[14]In general, Condition (11) is sufficient for a rotation to occur unless in the case of just two agent types, where (11), the rotation-property and stochastic dominance of the respective distribution functions are equivalent (see our working paper version).

[15]In the special case where $p(\cdot)$ is a Class II density and (11) holds for any $j > i$, (11) is known as the monotone likelihood property in mathematical statistics (see, e.g., Casella and Berger (2002)). In economic theory, monotone likelihood ratios are sometimes imposed by mechanism design or contract theory as *exogenous assumptions* on the ex ante type distribution, and therefore unrelated to this article.

If (11) holds for any $x, x' \in X$ with $x' > x$, then the ratio of market shares is a strictly monotonic function of x for any given pair of agents (i, j) with $j \rhd i$.[16] This implies that the rotations induced by $dx > 0$ are transitive: If $p(\cdot; x'')$ is a monotonic OR of $p(\cdot; x')$ and $p(\cdot; x')$ is a monotonic OR of $p(\cdot; x)$, then $p(\cdot; x'')$ must be a monotonic OR of $p(\cdot; x)$, too (similarly for IR). Thus, the relative market share $\frac{p(i;x)}{p(j;x)}$, $j \rhd i$, is strictly increasing (decreasing) in x with a monotonic OR (IR), meaning that market shares must be less and less equally (more and more equally) dispersed as x increases. That is, the inequality of $p(\cdot; x)$ must increase (decrease) over the entire parameter space X as measured by any Lorenz-consistent inequality measure.

Calculus Criteria I Our next result is helpful for identifying monotonic rotations in practice as it builds on the primitive formulation (8).

Theorem 3 (Monotonic Rotations) *Let $p(\cdot)$ be Class I or II, and $x_0, x \in X$. If*

$$\Delta_i(x) > (<) \Delta_j(x) \qquad \forall x \geq x_0 \textit{ and any } j \rhd i, \tag{12}$$

then $p(i; x)$ is a monotonic OR (IR) of $p(i; x_0)$ for any $x > x_0$.

Note that condition (12) is equivalent to

$$\frac{\partial}{\partial x}\left(\frac{p(i;x)}{p(j;x)}\right) > (<) 0 \qquad \forall x \geq x_0 \text{ and any } j \rhd i. \tag{12'}$$

Theorem 3 is useful in applications, because it says that whenever we can infer condition (12') from the equilibrium equation (6), $p(\cdot; x)$ must be a monotonic OR (or IR) of $p(\cdot; x_0)$ for any $x > x_0$.[17] Finally, we remark that if (12) or (12') hold with equality $\forall x$, then x is inequality preserving which, for completeness, is summarized next.

Corollary 1 *x is inequality preserving iff $\frac{\partial}{\partial x}\left(\frac{p(i;x)}{p(j;x)}\right) = 0$ $\forall i, j \in I$ and $\forall x \in X$.*

Calculus Criteria II Previously, we traced the existence of inequality effects or of rotations back to the direct-aggregative effect R. We now attempt to pursue this also in case of monotonic rotations, which delivers an alternative condition for their existence.

From (8), we observe that the indirect effect k_{ij} plays a moderating role on how Δ_j affects Δ_i. We now show that the indirect effect matters for certain quantitative aspects should a rotation occurs. If $k_{ij}(x_0) \geq (\leq) 1$ $\forall j \rhd i$, then we say that $k(x_0)$ is *uniformly larger (smaller) than one* at x_0. The following result states that if R is uniformly positive or negative, and k is uniformly larger or smaller than one, the market shares of either the winners or losers must evolve monotonically, depending on which out of four possible cases arises.

[16] If $p(\cdot; x)$ is Class II and condition (11) holds on X, this is equivalent to strict log-super(sub)modularity of $p(i; x)$, but not if $p(\cdot; x)$ is of Class I in view that $p(\cdot)$ is a step function. Thus, standard results from lattice theory cannot be applied to our setting. The working paper version provides a careful discussion of these technical aspects.

[17] In the working paper version we derive an alternative rotation condition that operates over differences instead of ratios.

Proposition 2 (Partially Monotonic Rotations) *Let $p(\cdot;x)$ be Class I, and suppose that R is uniformly positive (negative) at $x_0 \in X$. If $k(x_0)$ is uniformly larger than one, there is $\delta > 0$ such that*

$$p(i;x') > p(i;x) \quad \Rightarrow \quad \frac{p(i;x')}{p(i,x_0)} > (<) \frac{p(j;x')}{p(j,x_0)} \quad \forall j \rhd i \tag{13}$$

for any $x' \in (x_0, x_0 + \delta)$. If k is uniformly smaller than one, there is $\delta > 0$ such that

$$p(i;x') < p(i;x) \quad \Rightarrow \quad \frac{p(i;x')}{p(i,x_0)} > (<) \frac{p(j;x')}{p(j,x_0)} \quad \forall j \rhd i \tag{14}$$

for any $x' \in (x_0, x_0 + \delta)$.

In view of Definition 7, the conditions in Proposition 2 amount to *partially* monotonic rotations. If R is uniformly positive, such that an OR results (Theorem 2), and k is uniformly larger than one, (13) says that, among winning agents, the stronger an agent is (lower index i), the more the agent gains in relative terms. Equivalently, if agent i gains market share due to $dx > 0$ and $j \rhd i$, then the relative market share $\frac{p(i;x)}{p(j;x)}$ must have strictly increased. By contrast, if $k \leq 1$ uniformly, then (14) says that among the losing agents, the weaker an agent is the more she loses. The same logic applies "from the other side" if $R < 0$ uniformly, such that an IR results, and hence the weaker agents gain market shares while the stronger agents lose.

If k is uniformly *equal* to one, both statements of Proposition 2 apply. The key consequence, summarized next, is that then x must induce a monotonic rotation.

Theorem 4 *Let $p(\cdot)$ be a Class I or II density. If $k(x) = 1$ $\forall i, j \in [0,1]$ and any $x \in X$, and R is globally uniformly positive (negative), then $p(i;x)$ is a monotonic OR (IR) of $p(i;x_0)$ for any $x > x_0 \in X$.*

Intuitively, $k = 1$ means that the marginal costs and benefits of all agents respond equally sensitive, ceteris paribus, to changes in aspired market shares. Theorem 4 shows that in this case the mere sign-uniformity of the direct-aggregative effect suffices to assure that a monotonic rotation occurs.

3.4 Comparative-Statics of $T(x)$

By (9), $sign\, T'(x)$ matters for R, and thus for the resulting inequality effects. We now present a simple procedure to determine $sign\, T'(x)$, exploiting our reformulation as a competition for market shares.

- **Step I:** Fix an arbitrary agent i, and suppose that x and T are exogenous parameters. Then, for each i, (4) implicitly determines a function $p(i;T,x)$. Use the Implicit Function Theorem to determine the partial derivatives $p_x(i;T,x)$ and $p_T(i;T,x)$.

- **Step II:** Define $G(T,x) \equiv \int p(i;T,x)di$. Use Step I to determine $G_T(T,x)$ and $G_x(T,x)$. Use the equilibrium equation $G(T,x) = 1$ to determine $T'(x)$.

We illustrate this procedure by proving the following result.

Lemma 3 *Let Assumptions 1-2 be satisfied,* $g_x(i) > 0$ *and* $\varphi_x(i) = 0\ \forall i$. *Then* $T'(x) > 0$.

Proof: Consider an arbitrary agent i. **Step I:** By (A1), equation (4) must have a unique solution $p(i;T,x)$ for any given T,x. Quasiconcavity (A1) and $g_x > 0$ further imply that $p_x(i;T,x) > 0$. Likewise, quasiconcavity and the fact that $g_T(i) < \varphi_T(i)$ by (A2), together assure that $p_T(i;T,x) < 0$. **Step II:** For $G(T,x) \equiv \int p(i;T,x)di$ it follows from Step I that $G_x(T,x) > 0$ and $G_T(T,x) < 0$. Based on the equilibrium equation $G(T,x) = 1$, the Implicit Function Theorem and Step II implies that $T'(x) > 0$.

In the literature on aggregative games, the comparative-statics of aggregate quantities is a central question, and creative ways have been identified to establish monotone comparative-statics of these aggregates.[18] While our main contribution – systematically studying the array of inequality effects in models with an aggregative structure – is mostly unrelated to that literature,[19] the above procedure adds a simple way to determine $sign\, T'(x)$.

3.5 Further Remarks

Theorems 1, 2 and 4 provide powerful conditions for studying market inequality because R_{ij} is a local condition which nevertheless is informative about a global property of an equilibrium distribution $p(\cdot)$. With respect to applicability, we shall see that the uniformity requirement on R is met by most of our applications in Section 5 (given that $R \neq 0$). Moreover, Theorems 1 and 2 are analytically useful in specific applications, because we do not need to explicitly solve the equilibrium equation to calculate $sign\,(R_{ij})$. This allows us to study, e.g., how $sign\,(R_{ij})$ depends on intrinsic properties of a model (see Section 5).

4 Inequality Analysis

We now use the abstract properties from Section 3 to identify more tangible properties of the equilibrium equation (4) or the payoff function $\Pi(i)$ that are associated with specific inequality effects.

4.1 Multiplicative Separability

Our first result presents an equivalent condition to $R = 0$. Expression (9) shows that $R_{ij} = 0$ iff $A(i) - A(j) = 0$. The latter difference can be decomposed in a "benefit" and "cost" side:

$$\begin{aligned} A(i) - A(j) &= \left(\frac{g_T(i)}{g(i)} - \frac{g_T(j)}{g(j)}\right) T'(x) + \left(\frac{g_x(i)}{g(i)} - \frac{g_x(j)}{g(j)}\right) \qquad \text{"benefit side"} \\ &+ \left(\frac{\varphi_T(j)}{\varphi(j)} - \frac{\varphi_T(i)}{\varphi(i)}\right) T'(x) + \left(\frac{\varphi_x(j)}{\varphi(j)} - \frac{\varphi_x(i)}{\varphi(i)}\right) \qquad \text{"cost side"} \end{aligned} \tag{15}$$

[18] See, e.g., Corchon, 1994; Cornes and Hartley, 2012; Acemoglu and Jensen, 2010, 2013; Camacho et al., 2018. Further, Jensen (2018); Corchón (2020) provide nice surveys.

[19] Perhaps most closely related is Acemoglu and Jensen (2013), who utilize the aggregative structure to obtain comparative-static predictions of the aggregate and extremal efforts under general conditions, and sometimes of certain individual strategies. In contrast to what we do, their paper does not study inequality effects of market shares or payoffs.

Decomposition (15) shows that $R_{ij} = 0$ if all four brackets are zero. The terms in these brackets capture whether T and x affect marginal benefits or costs differentially for different agents. Considering the first bracket, we note that this bracket is zero if $\frac{g_T(i)}{g(i)} = \frac{g_T(j)}{g(j)}$, i.e., if T affects agents i and j by *equal proportions*. It immediately follows that if x and T affect marginal benefits and costs of all agents by *equal proportions*, then x must be inequality preserving. The following result builds on this observation and the fact that the principle of equal proportions is equivalent to multiplicative separability.[20]

Definition 10 (Multiplicative Separability) *A function $f(i, p, T, x)$ is multiplicatively separable in (i, p) and (T, x) if there are functions $u(i, p)$, $v(T, x)$ such that $f(i, p, T, x) \equiv u(i, p)v(T, x)$.*

Theorem 5 *The parameter x is inequality preserving iff the function $\frac{g(i,p,T;x)}{\varphi(i,p,T;x)}$ is multiplicatively separable in (i, p) and (T, x).*

Theorem 5 is useful because it can be easily checked from the optimality condition (4). To see the intuition, note that $\frac{g(i)}{\varphi(i)} = \frac{g(j)}{\varphi(j)}$ for any two agents by equilibrium forces. Thus, if the ratio $\frac{g(i)}{\varphi(i)}$ verifies multiplicative separability, then this ratio must change by equal proportion for each agent, which means that no agent can secure a systematic advantage due to $dx \neq 0$ in equilibrium, resulting in a stable dispersion of market shares.

4.2 Level Variables and Neutral Costs

Decomposition (15) shows that the cost and benefit sides may contribute separately to the possible inequality effects. Specifically, if either the benefit or the cost side verifies multiplicative separability, the side has a "neutral" effect on market inequality. This is captured by the following definition.

Definition 11 (Level Variables and Neutral Costs) *The market condition x is a **level variable** if x only enters $g(i)$ ($\varphi_x(i) = 0\ \forall i$), and the function $g(i)$ is multiplicative separable in (i, p) and (T, x). Further, we say that costs $\varphi(i)$ are **neutral** if the function $\varphi(i)$ is multiplicative separable in (i, p) and (T, x).*

A simple example for a level variable is $g(i) = V(T; x)$, which also shows up in our later applications. Also note that, by definition, costs are neutral if x does not enter $\varphi(i)$ but $\varphi(i)$ is multiplicatively separable in (i, p) and T. If $p(i) = t(i)/T$ and x does not enter $\varphi(i)$, then the neutral costs property equivalently means that the costs $\Phi(i, t)$ of the untransformed payoff function (1) must be a power function, as the following result shows.

Lemma 4 *If $p(i) = t(i)/T$ and $\varphi_x(i) = 0\ \forall x, i$, costs are neutral iff $\Phi(i, t) = c(i)t^\gamma$.*

We can now derive more specific inequality results, that will also matter for our applications. The first result states that a level variable induces inequality effects iff costs are not neutral.

[20] We provide a short proof of this fact in the working paper version.

Proposition 3 *The level variable x is inequality preserving if costs $\varphi(i)$ are neutral. Conversely, if x is inequality preserving and $T'(x) \neq 0$ $\forall x \in X$, then costs $\varphi(i)$ must be neutral.*

We next show that the cost function determines key properties of the inequality effects that emerge for $dx > 0$ if x is a level variable. Let

$$\theta(i) \equiv \frac{\varphi_T(i)T}{\varphi(i)} \tag{16}$$

denote the elasticity function of marginal costs. If x is a level variable, it is is easily checked that costs are neutral iff $\theta(i) = \theta(j)$ for all i, j and any T. Moreover, because $sign\, R_{ij} = sign\,(\theta(j) - \theta(i))\, T'(x)$, the elasticity function $\theta(\cdot)$ then affects whether a rotation occurs.

Proposition 4 *If $T'(x) > 0$, $p(\cdot)$ is Class I, and $\theta(i) < (>)\theta(j), \forall j \rhd i$, then the level variable x induces an OR (IR) of $p(\cdot)$.*

The result follows directly from Theorem 2, and it also applies for $T'(x) < 0$ if "OR" and "IR" are interchanged. The intuition of Proposition 4 is as follows. An increase in x induces the same incentive for all agents to aspire for a larger market shares, while the costs associated with such changes may be differentially sensitive. If $\theta(i) < \theta(j)$ for $j \rhd i$, then agent i can adjust better to the new situation, as her marginal costs increase at a slower pace. If such a ranking applies for any two different agent types, the adjustment is easiest for the agents with the largest market shares, explaining why an OR results.

If $p(i) = t(i)/T$, additional insights emerge. First, Proposition 4 can be equivalently expressed in the elasticity of the primitive marginal cost function $\Phi_t(i,t) \equiv h(i,t)$. Let $\psi(i) \equiv h_t(i,t)t/h(i,t)$ denote the t-elasticity of marginal costs $h(i,t)$. It is easy to check that $\theta(i) = \psi(i) + 1$, which shows that the elasticity condition in Proposition 4 can be restated as $\psi(i) < (>)\psi(j)$. Second, if $g(i)$ additionally is a power function of p, the rotations identified by Proposition 4 must be partially monotonic in the sense of Proposition 2.

Corollary 2 *If $p(\cdot)$ is Class I, $p(i) = t(i)/T$, $g(i) = z(i)p(i)^\alpha u(T;x)$, where $\alpha \in \mathbb{R}$ is a constant, $T'(x) > 0$, and $\psi(i) < (>)\psi(j)$ $\forall j \rhd i$, then the level variable x induces a partially monotonic OR (IR) with property* (13) *(*(14)*).*

4.3 Power Functions

Our next set of results is centered around power functions, which often arise in applications. Lemma 4 or Corollary 2 exemplified that power functions play a role for the inequality effects. More generally, a key observation is that power functions pin down the indirect effect: $k_{ij} = 1$ uniformly for any $x \in X$ *iff* the ratio $\frac{\varphi(i)}{g(i)}$ has a power function form:

$$\frac{\varphi(i,p,T;x)}{g(i,p,T;x)} \equiv z(i,T;x)p^{\xi(T;x)}, \qquad p, T > 0. \tag{17}$$

Proposition 5 *If* (17) *holds, then* $k(x) = 1$ *uniformly* $\forall x \in X$*. Thus, if additionally* R *is globally uniformly positive (negative), then* $p(\cdot; x')$ *is a monotonic OR (IR) of* $p(\cdot; x)$ $\forall x' > x$*. Conversely, if* $k(x) = 1$ *uniformly for any given* i, j, T, x*, then* (17) *holds.*

As the power function property is invariant to integration, a sufficient condition for (17) is that the costs and benefits in $\Pi(\cdot)$ are power functions of p with i-independent exponents.

4.4 Market Inequality in Payoffs, Revenues or Expenditures

Our inequality results so far focused on market shares $p(\cdot)$. We now sound out the relation between the dispersion of $p(\cdot)$ and other key quantities, such as payoffs. While not much can be said about these relations in general, definite patterns emerge under more special circumstances. Let $s(i) \equiv \Pi(i)/\int_I \Pi(s)ds$ denote the *payoff share* earned by agent i. Likewise,

$$b(i) \equiv \frac{p(i)V(i,p(i),T;x)}{\int_I p(s)V(i,p(s),T;x)ds}, \qquad e(i) \equiv \frac{\Phi(i,p(i),T)}{\int_I \Phi(s,p(s),T)ds}$$

denote the *benefit* and *expenditure shares*, respectively. Our first result relates the possible inequality effects of these shares to multiplicative separability of the payoff function (3).

Proposition 6 *If benefits* $pV(i,p,T;x)$ *and costs* $\Phi(i,T;x)$ *are multiplicatively separable in* (i,p) *and* (T,x)*, then market shares* $p(\cdot)$*, benefit shares* $b(\cdot)$ *and expenditure shares* $e(\cdot)$ *all are invariant to* x*.*

The main reason for Proposition 6 is that payoffs are of the form "Benefits" minus "Costs", and multiplicative separability is preserved under differentiation wrt $p(i)$.

Power Functions The relation between the various market shares tightens if benefits and costs have power function representations. Let $\Pi(i)$ be of the form

$$\Pi(i) = p(i)^{\alpha}\hat{g}(i,T;x) - p(i)^{\beta}\hat{\varphi}(i,T;x), \tag{18}$$

where generally $\alpha = \alpha(T;x)$ and $\beta = \beta(T;x)$ with $\beta > \alpha > 0$ for any given $(T;x)$.[21] Using short-hand notation for $\hat{g}(\cdot)$ and $\hat{\varphi}(\cdot)$ in (18), the equilibrium condition (6) can be stated as

$$\frac{p(i)}{p(j)} = \left(\frac{\hat{g}(i)}{\hat{g}(j)}\frac{\hat{\varphi}(j)}{\hat{\varphi}(i)}\right)^{\frac{1}{\beta-\alpha}}. \tag{19}$$

Moreover, using (19) in (18) implies

$$\frac{s(i)}{s(j)} = \left(\frac{p(i)}{p(j)}\right)^{\alpha}\frac{\hat{g}(i)}{\hat{g}(j)} = \left(\frac{p(i)}{p(j)}\right)^{\beta}\frac{\hat{\varphi}(i)}{\hat{\varphi}(j)} = \left(\left(\frac{\hat{g}(i)}{\hat{g}(j)}\right)^{\beta}\left(\frac{\hat{\varphi}(j)}{\hat{\varphi}(i)}\right)^{\alpha}\right)^{\frac{1}{\beta-\alpha}}. \tag{20}$$

The first two equations in (20) imply that $\frac{s(i)}{s(j)} = \frac{b(i)}{b(j)} = \frac{e(i)}{e(j)}$. Noting that all dispersions $p(\cdot)$, $s(\cdot)$, $b(\cdot)$ and $e(\cdot)$ are i-densities with the same equivalence classes $[i]$, the following Lemma

[21] The latter requirement follows from quasiconcavity. Note that the formulation in (18) includes the case, where, e.g., $\alpha > 0$ simply is an exogenous constant.

is helpful for clarifying the relation between them.

Lemma 5 *Let $q(\cdot)$ and $r(\cdot)$ be two densities with identical support $[0,1]$. Then $q(i) = r(i)\ \forall i$ iff $\frac{q(i)}{q(j)} = \frac{r(i)}{r(j)}\ \forall i, j$. If, in addition, $q(\cdot)$ and $r(\cdot)$ are two SSD densities with equal equivalence classes $[i]$, and $\frac{q(i)}{q(j)} > (<) \frac{r(i)}{r(j)}\ \forall j \rhd i$, then $q(\cdot)$ is a monotonic OR (IR) of $r(\cdot)$.*

To avoid misunderstanding, the statement "$q(\cdot)$ is an OR of $r(\cdot)$" means that the two densities $q(\cdot)$ and $r(\cdot)$ are ranked by the OR criterion analogously to Definition 7. That is, $q(\cdot)$ crosses $r(\cdot)$ once from above. This leads to the following result.

Proposition 7 *If $\Pi(i)$ is of type (18), then $s(i) = b(i) = e(i)\ \forall i$, i.e., payoff, benefit and expenditure shares always coincide.*

By Proposition 7, the agents with the largest benefits (or payoffs) also incur the highest expenditures, despite their possible advantages in the ex ante costs. In addition, the proposition predicts that benefit and expenditure shares move in lockstep, which is a common observation across different markets (Jones, 1990). The relation between market shares and the other shares is more intricate even in the power function case. However, as we shall see later applications, Lemma 5 together with (19) and (20) frequently allow us to disentangle the relation between these shares as well.

4.5 Procedural Remarks

As their proofs reveal, the previous results essentially all follow from applying the same procedure that is suitable for other situations, too. Its steps are displayed in Figure 4. It is

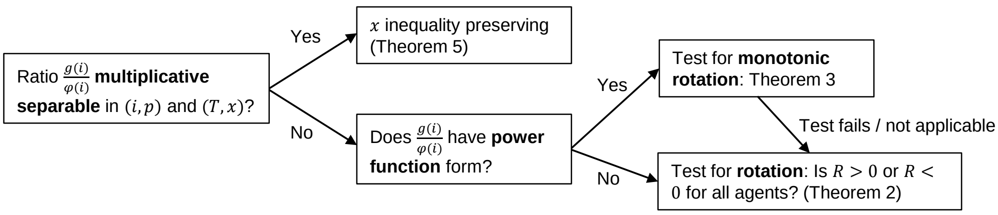

Figure 4: Procedure

important to mention that even if market shares are stable with respect to x, there still can be absolute *absolute* inequality effects in payoffs, benefits or expenditures as x changes. For example, the benefit share $b(\cdot)$ can be stable while the level of individual benefits, $p(i)V(i)$, increase in x. In such a case, the absolute gap $p(i)V(i) - p(j)V(j)$ must increase in x for any $j \rhd i$ (see Lemma 2). Such effects have sometimes been called Matthew effects or the "rich gets richer".

5 Applications

As a first step in taking our general results to applications, we state various examples that allow for representation as competition for market shares (Section 5.1). We then split the inequality analysis between models featuring *symmetric* competition (Section 5.2), where all agents earn the same equilibrium value per unit of market shares, and those featuring *asymmetric* valuations (Sections 5.3-5.4). In what follows, we adapt the standard convention of indicating different agents by using subscripts, e.g., we write Π_i instead of $\Pi(i)$.[22]

5.1 Competition for Market Shares: Examples

We show that three different types of competition – competition for prizes, perfect competition and monopolistic competition – can be represented as a competition for market shares. A central insight is that these three distinct models of competition share the same formal structure, allowing for a unified inequality analysis.

Competition for Prize A first class of models that fits our framework can be described as a "competition for prize". A simple example are *fixed-prize contests*, where different agents compete in efforts $t_i \geq 0$ to seize a single prize of a value $V > 0$ common to all agents. While the literature on contests is large, papers that study inequality in contests are rare (see Konrad, 2009). Other examples suggest that the prize value $V(\cdot)$ is agent-specific or endogenous. For example, litigation expenditures (Posner, 1992), salary negotiations (Amegashie, 1999), or (money) efforts invested to obtain a monopoly franchise (Chung, 1996) can influence the terminal value $V(\cdot)$ earned by the winning agent.

We formalize these notions as follows. Each agent $i \in [0, 1]$ chooses an effort $t_i \geq 0$ that influences her chance of winning a prize which, in general, is determined by a *value function* $V(i, t_i, T; x)$. The winning chance of agent i is $p_i = t_i/T$, where $T = \int t_s ds$, and p_i thus verifies the formal property of a market share.[23] Assuming a general cost function $\Phi(i, t_i)$, this model yields a competition for market shares with payoffs

$$\Pi_i = p_i V\left(i, p_i T, T; x\right) - \Phi\left(i, p_i T\right). \tag{21}$$

A key property that generally distinguishes contests from market-based competition, is that p_i depends only on *relative* efforts in the former, while frequently the absolute values of prices or quantities matter for the latter.[24]

Perfect Competition A central example for a model of market-based competition is perfect competition. Each firm $i \in [0, 1]$ produces a quantity $q_i \geq 0$ of a homogeneous good according to cost function $\Phi(i, q_i)$, taking the price P as given. Market demand is given by a function $P(T; x) > 0$, where $T = \int q_i di$ is aggregate supply, and $P_T < 0$ by the Law of

[22] This notational change is meant to increase legibility. When developing our formal results in Sections 3 - 4 the previous notation was more convenient to clarify, e.g., that $\Pi(\cdot)$ is a function of the agent index i and other variables.

[23] The simple formulation $p_i = t_i/T$ is wlog in the followings sense. If the "success function" $p(t_i, T)$ is strictly increasing in t_i, zero-homogeneous in (t_i, T), and $\int p(t_i, \int t_s ds) di = 1$, then the only function that satisfies these conditions must be of the form $p_i = t_i/T$ (Hefti and Lareida, 2020).

[24] By (21), we assume that agents take the equilibrium aggregate T as given when choosing their efforts. In Appendix B.5, we show how to adjust (21) if finitely many different agents endogenize the effects of their effort choices on T, thereby embedding the notion of Nash equilibrium in our setting.

Demand.[25] Let $p_i \equiv \frac{Pq_i}{\int Pq_s ds}$ denote firm i's market share of total consumption expenditures. Because $p_i = q_i/T$, condition (2) is verified, and restating the payoff as a competition for market shares (in terms of consumption expenditures) yields

$$\Pi_i = Pq_i - \Phi(i, q_i) = p_i P(T; x)T - \Phi(i, p_i T). \tag{22}$$

We think of q_i as produced with a production function $q_i = f_i(\frac{\rho}{\alpha_i} y)$, where $\rho > 0$ is a common and $\alpha_i > 0$ and individual productivity parameter, and $y = (y_1, ..., y_K)$ an input vector. Under the common assumption that $f_i(y)$ is a *homogeneous function* of, say, degree $1/\gamma_i$, and y is acquired in competitive markets, the corresponding cost function must be a *power function* $\Phi(i, q_i) = \frac{\alpha_i}{\rho} q_i^{\gamma_i} h_i(w, 1)$.[26] Thus, setting $c_i \equiv \alpha_i h_i(w, 1)$ and using the identity $q_i = p_i T$, we obtain

$$\Phi(i, p_i T) = \frac{c_i}{\rho}(p_i T)^{\gamma_i}. \tag{23}$$

An interesting aspect of (23) is that it separates between ex ante firm heterogeneity due to differences in *productive efficiency* c_i/ρ or due to different *returns to scale* γ_i. As we shall see, the precise source of the ex ante firm heterogeneity can be decisive for the inequality effects, and the simple structure of (23), at the very least, helps to make this most evident.

Once stated as a competition for market share, it becomes obvious that we can view perfect competition (22) as a *common-prize contest with endogenous valuation* (21) (and vice-versa) by defining the value function as $V(\cdot) \equiv P(T; x)T$. This is remarkable as there is no general analogue in market-based competition to the relative nature of competition characteristic for contest-like settings.

On this matter, we remark that cost function (23) makes sense in contests (21), too. For example, if $\gamma_i = \gamma\ \forall i$, then γ is a measure of how "noisy" the contest is, i.e., of how easy it is for individual agents to influence their chances of success (Hefti, 2018).

Monopolistic Competition Our third application is monopolistic competition (Dixit and Stiglitz, 1977). Each firm i supplies a quantity $q_i \geq 0$ of a differentiated product at a monopolistically chosen price P_i. There is a continuum of consumers, indexed by $\iota \in [0, 1]$, each endowed with a utility function

$$U(\iota) = \int_0^1 r_s q_s(\iota)^\sigma ds, \qquad \iota \in [0, 1], \tag{24}$$

where $s \in [0, 1]$ denotes a product, and $r_s > 0$ measures the importance of product s (e.g., quality), $q_s(\iota) \geq 0$ is the respective quantity demanded by consumer ι, and $\sigma \in (0, 1)$ the elasticity of substitution. Consumer ι has disposable income $I(\iota) > 0$, and chooses each $q_s(\iota)$ to maximize (24), subject to $\int P_s q_s(\iota) ds = I(\iota)$. For $\eta \equiv \frac{1}{1-\sigma} > 1$, this optimization problem

[25] Such a demand function can further be microfounded, e.g., by a partial equilibrium setting, which we pursue in the working paper version. Then, the demand shifter x can, e.g., represent a parameter related to consumer utility.

[26] $h_i(w, 1)$ solves $\min_y w \cdot y$, s.t. $f_i(y) = 1$.

leads to aggregate demand

$$q_i = \frac{I r_i^\eta P_i^{-\eta}}{\int r_s^\eta P_s^{1-\eta} ds}, \qquad I \equiv \int I(\iota) d\iota > 0 \tag{25}$$

for each product i. As with perfect competition, we let $p_i = \frac{P_i q_i}{\int P_s q_s ds}$, amount to firm i's market share of total consumption expenditures.

To state monopolistic competition as a competition for market shares, note from (25) that, by setting its price P_i, each firm also chooses its quantity q_i. For any (integrable) price profile P, we define the aggregator function $Z(P) = \int r_s^\eta P_s^{1-\eta} ds$, where the value $T = Z(P)$ can be interpreted as a preference-weighted inverse price index.[27] Quantity q_i is produced with a homogeneous production function, such that costs are (23). This includes constant marginal costs ($\gamma_i = 1$), which is the most common assumption.[28] These definitions and (25) yield the payoff function

$$\Pi_i = P_i q_i - c_i q_i^{\gamma_i} = I p_i - w_i I^{\gamma_i} p_i^{\frac{\gamma_i \eta}{\eta - 1}} T^{\frac{\gamma_i}{\eta - 1}}, \qquad w_i \equiv \frac{c_i}{\rho} r_i^{-\frac{\gamma_i \eta}{\eta - 1}}. \tag{26}$$

The formulation as a competition for market shares shows that monopolistic competition is akin to a *contest for total income*. Nevertheless, (26) differs from contests (21) or from perfect competition (22), as the costs in (26) are not of the form $\Phi(i, p_i T)$. This reflects that in monopolistic competition the aggregator function $Z(\cdot)$ does not amount to the sum of "actions" (i.e., prices).

5.2 Common Valuations

We now study a class of models where the agents earn an identical benefit per unit of market share, such that payoffs are of the form

$$\Pi(i) = p(i) V(T; x) - \Phi(i, p(i), T; x), \tag{27}$$

and hence $g(i) = V(T; x)$. This represents a situation where heterogeneous agents, in terms of ex ante costs, compete for an object of a *common but possibly endogenous valuation.* Throughout our analysis, we suppose that (27) verifies Assumption 1 and let ex ante agent heterogeneity to be such that $p(\cdot)$ is a Class I or II density $\forall x \in X$.

We recognize perfect competition (22) and monopolistic competition (26) as variants of (27), and so is (21) in case of a contest with common prize function (including standard fixed-prize contests). These models therefore allow for a unified inequality analysis. We begin this analysis by clarifying the relation between market shares and other quantities.

Proposition 8 *If a competition for market shares has payoffs of the form* (27)*, then:*

i) Market and benefit shares always coincide ($p_i = b_i \ \forall i$)*.*

[27] To see that $(\{\Pi_i\}, Z(\cdot))$ indeed constitute a competition for market shares, note that p_i can be equivalently stated as $p_i = \frac{r_i^\eta P_i^{1-\eta}}{T}$, which shows that (2) is verified.

[28] A prominent example is Melitz (2003) and subsequent applications in international trade.

ii) In addition, market shares, expenditure shares and payoff shares coincide ($p_i = e_i = s_i$ $\forall i$) *if costs are a power function of the form* $\Phi(i) = p_i^{\beta(T;x)} h(i, T; x)$.

The main reason why the symmetric competition model yields a strong connection between the various market shares essentially goes back to the power function arguments from Section 4.4, by noting that $g(i)$ naturally satisfies the power function property here.

We now proceed as follows. We first present general inequality results based on payoff (27), where we distinguish between benefit- and cost-side conditions x (Section 5.2.1). We thereafter refine these insights in the context of our examples (Sections 5.2.2 - 5.2.3). Section 5.2.4 provides a comprehensive intuition, and relates some of our predictions to the literature.

5.2.1 General Inequality Analysis

In general, the market condition x could enter valuations $V(\cdot)$ or costs $\Phi(x)$, and we distinguish between both cases to structure our analysis. If x only enters $V(\cdot)$, x must be a *level variable*. For example, x could represent a demand shifter in $P(T; x)$ from (22), or a prize shifter in $V(T; x)$ of a contest. Another interpretation is that $dx > 0$ captures an increase in a quantity or sales tax, say, in perfect competition – the former yields $P(T; x) = P(T) - x$ and the latter $P(T; x) = (1 - x)P(T)$. For definiteness, we assume that $V_x(T; x) > 0$ if x is a level variable. Then, the procedure from Section 3.4 and Assumption 1 assure that $T'(x) > 0$.[29]

We say that x is a *cost-side condition* if x only affects the costs in (27). For example, x could quantify a common efficiency level, such as ρ in (23), or a general cost shifter reflecting, e.g., deflated factor prices. If x is a cost-side condition we assume, again for definiteness, that $\varphi_x(i) < 0$ $\forall i$. Then, much like a level variable, $dx > 0$ incentivizes all agents to aspire for a larger market share, implying that $T'(x) > 0$.

Theorem 5 showed that x is inequality preserving if (27) is multiplicatively separable in (i, p) and (T, x). The following proposition evaluates this result in the current context.

Corollary 3 *If x is a level variable with $V_x(T; x) > 0$, then x is inequality preserving iff costs are neutral. If x is a cost-side condition, then x is inequality preserving iff costs are multiplicatively separable in (i, p) and (T, x).*

Note that the previous result applies similarly for benefit shares, and also for payoff and expenditure shares if condition ii) from Proposition 8 is verified. This also holds analogously for the next result, which shows that x must induce a rotation whenever agents are ordered according to the cost elasticities θ_i from (16).

Corollary 4 *Let $p(\cdot)$ be Class I. If x only affects $V(\cdot)$ and $V_x(T; x) > 0$, or if x is a cost-side condition, then $dx > 0$ induces an OR (IR) of $p(\cdot)$ if $\theta_i < (>) \theta_j$ $\forall j \rhd i$ in equilibrium.*

The main conclusion to draw from the above two results is that the inequality effects depend entirely on the cost function if x only affects $V(\cdot)$ (is a level variable) or a cost-side condition;

[29] If instead $V_x(T; x) < 0$, e.g., as in the tax example, this simply reverts the inequality predictions ("OR" becomes "IR" etc).

things like the curvature or slope of $V(T;x)$, e.g., the price elasticity of the demand function $P(T;x)$, play no role for the dispersion of market shares.

5.2.2 Returns to Scale and Market Inequality

By Corollaries 3 - 4, the cost elasticities are important for the inequality effects. These elasticities have an immediate connection to the *scale effects* on which, figuratively spoken, the "production" of market shares is based on. Therefore, the returns to scale ought to play a major role for the possible inequality effects induced by x.

Observe from (27) that if $p_i = t_i/T$, like in perfect competition or the contest setting, costs are neutral *iff* $\Phi(i)$ is a power function (23) with $\gamma_i = \gamma\ \forall i$, i.e., iff all firms are subject to *identical scale effects.* The same goes for monopolistic competition (26). The following two results essentially tailor Corollaries 3 - 4 to this situation, and has implications for several examples of x.

Corollary 5 *Suppose that costs are of type* (23)*, and let x either only affect $V(\cdot)$ with $V_x(T;x) > 0$ or be a cost-side condition with $\varphi(i) \equiv \hat{\varphi}(i, p_i, T)u(x)$. Then, x is inequality preserving iff $\gamma_i = \gamma\ \forall i$. Further, if $p(\cdot)$ is Class I, $\gamma_i < \gamma_j$ and $p_i > p_j\ \forall j \rhd i$, then $dx > 0$ induces a partially monotonic OR in the sense of* (13)*.*

This result also applies to payoff, benefit and expenditure shares due to the power function structure of $\Pi(i)$. Corollary 5 implies that common parameters, such as efficiency level ρ, a common quality level $r_i = r\ \forall i$ in monopolistic demand, a demand shifter in $P(T;x)$, or a sales tax have no inequality effects *iff* all firms are subject to *identical* scale effects γ. By contrast, if firms differ in their returns to scale, these variables necessarily induce inequality effects, which must be rotations whenever firms are ordered according to scale effects.

In applications with monopolistic competition, constant marginal costs is the most common assumption, which means that $\gamma_i = 1\ \forall i$. This is a special case where all firms have identical scale effects. More generally, if all firms have identical scale effects ($\gamma_i = \gamma\ \forall i$), then the scale γ itself must affect market inequality by Corollary 3 ($x = \gamma$), because multiplicative separability is violated. Intuitively, stronger scale effects (lower γ) mean that marginal costs respond less sensitively to an increase in market shares, which is more beneficial for firms operating at a higher level. Thus, we would expect industries with *stronger scale effects*, ceteris paribus, to display *more market inequality.*

The next result confirms this intuition, and applies to any competition with common valuations and $p_i = t_i/T$ (as in perfect competition or contests), as well as to monopolistic competition (which violates $p_i = t_i/T$). Let $\Phi(i)$ be a power function of type (23) with $\gamma_i = \gamma\ \forall i$ and $c_i < c_j$ iff $j \rhd i$. In case of monopolistic competition, we additionally assume that $r_i \geq r_j$ for any $i < j$, such that the agents are sorted according to costs also in this case. As before, the following result is valid for payoff, benefit and expenditure shares.

Corollary 6 (Scale Effects) *If $\gamma_i = \gamma\ \forall i$, then $d\gamma > 0$ induces a monotonic IR of $p(\cdot)$.*

5.2.3 Monopolistic Competition

We now provide a selection of inequality results for monopolistic competition that we deem important, also relative to the existing literature. We let $c_i \leq c_j$ and $r_i \geq r_j$ whenever $i < j$ for the remainder of this section.

First, Corollary 6 shows the inequality implications that different levels of common scale effects have. With respect to the literature, this result reveals that the standard assumption of constant marginal costs ($\gamma_i = 1\ \forall i$) induces the most extreme market inequality among all technologies with non-increasing returns to scale, ceteris paribus.

We next concentrate on the inequality effects implied by "love-of-variety" η and total income I, as these two aspects received attention in other studies.

Market Power Effects The parameter η pins down the elasticity of substitution, where a larger value of η means that products are stronger substitutes – or that there is a weaker "love-of-variety". In this respect, the literature frequently interpreted η as a parameter of *market power*, where a larger value of η means less firm-side market power.[30] The following proposition shows how a change in market power η affects the firm-side market inequality under the conventional assumption of identical scale effects.

Proposition 9 (Market-Power Effect) *Let $\gamma_i = \gamma \geq 1\ \forall i$. Then $d\eta > 0$ induces a monotonic OR of $p(\cdot)$.*

This result, and all subsequent ones, holds for payoff, expenditure and benefit shares as well. Proposition 9 shows that *less* market power implies *more* concentrated markets. Intuitively, competition is intensified if η increases and products become stronger substitutes. While this implies that all firms want to save costs, the equilibrium response depends on the current market shares. As (26) shows, a larger value of η formally has the same effects on marginal costs as a smaller γ (i.e., weaker scale effects). Thus, firms with larger current market shares, due to favorable ex ante conditions, can adopt better to a more competitive market.

The above market power argument relies on the "firm's ability to make the price" (Tirole, 1988), which depends on η and becomes relevant through monopolistic price-setting. Nevertheless, we emphasize that *none* of the inequality effects we identified earlier in monopolistic competition actually hinge on monopolistic behavior itself: in the working paper version we prove that identical inequality effects arise with price-taking firms.

Income Effects We now consider the inequality effects of an increasing total income I. Note from (26) that I enters costs and benefits, meaning that the inequality effects induced by I are not directly evident from our previous result. Our key observation is that changes in income induces inequality effects *iff* the firms differ in their scale effects. We discuss the intuition of this central result in Section 5.2.4 in a broader context.

[30] For example, with ex ante homogeneous firms (such that $p_i = 1\ \forall i$), the unique equilibrium price monotonically approaches the perfectly competitive price whenever $\eta \to \infty$.

Proposition 10 (Income Effects) *Total income I is inequality preserving iff $\gamma_i = \gamma\ \forall i$. Further, if $\gamma_j > \gamma_i$ and $p_j < p_i\ \forall j \rhd i$, then $dI > 0$ induces a monotonic OR of $p(\cdot)$, jointly with growing quantities $dq_i > 0\ \forall i$.*

Price Dispersion Our last inequality result pertains to an aspect that frequently shows up in data: a non-uniform *price dispersion.* With ex ante heterogeneous firms, the monopolistic competition model generally predicts heterogeneous prices - but to what extent does this price dispersion reflect common market conditions? As we already understand how market shares evolve in x, a first step is to study whether the price dispersion evolves in a similar manner. To address this question, we define $\pi_i \equiv \frac{P_i}{\int P_s ds}$ as firm i's price share and, noting that $\frac{\pi_i}{\pi_j} = \frac{P_i}{P_j}$, interpret the density $\pi(\cdot)$ as the *dispersion of relative prices.* Thus, by Corollary 1, the price dispersion changes iff relative prices $\frac{P_i}{P_j}$ change. The following proposition shows that the price dispersion changes iff market shares $p(\cdot)$ change. In addition, we exemplify how the price dispersion changes in income I or productivity ρ if firms differ in their scale effects, while offering a common quality ($r_i = r\ \forall i$).[31]

Proposition 11 (Price Dispersion) *The price dispersion $\pi(\cdot)$ is increasing over firm types $[i]$, and is invariant to any exogenous parameter x iff x is inequality preserving on $p(\cdot)$. If $\gamma_j > \gamma_i$ and $p_j < p_i\ \forall j \rhd i$, then $dI > 0$ or $d\rho > 0$ induce a counter-clockwise rotation of $\pi(\cdot)$, hence leading to a steeper price dispersion.*

The price dispersion is increasing in firm index i, reflecting that higher-cost firms set higher equilibrium prices. The second result of Proposition 11 implies that the price inequality evolves analogously to market share inequality if firms differ in their scale effects and I or ρ change. Thus, firms with favorable scale effects ($\gamma_i < \gamma_j$) can adjust their equilibrium prices in a way that allows them to attract more demand, either by decreasing their prices more or increasing them less compared to disadvantaged firms.

5.2.4 Main Intuition and Related Literature

Can certain agents take advantage of commonly improving market conditions relative to their competitors? If the competition between ex ante heterogeneous agents is *symmetric*, where agents with different cost functions compete for an object of common value, our analysis showed that the answer to this question is non-trivially determined by the cost side. Specifically, whether there is agent heterogeneity in the returns to scale associated with obtaining a certain market share turned out to be decisive for inequality effects to arise. Changes in market conditions, such as a reduction or elimination of a sales tax, or a common shift in productive efficiency due to an industry-level innovation leave the dispersion of market shares unaffected *iff* production is subject to exactly the same scale effects for all firms. This pattern applies more generally whenever x is a market condition that only enters the valuation per unit of market share $V(\cdot)$ or the cost function of each firm in a multiplicatively separable way.

[31] The working paper analyzes the price dispersion for more complex cases.

Intuition To see the main reason for this prediction, consider for the sake of illustration a market condition x that only enters $V(\cdot)$ (i.e., x is a level variable), and assume power-law cost functions (23), where firms can differ in their productive efficiency c_i or their returns to scale γ_i. Because competition is symmetric, equilibrium forces equate marginal costs across all agents. Therefore, marginal costs must adjust by the same rate in equilibrium for all agents once x changes. This competitive property explains why the scale effects are decisive for the inequality effects caused by x.

With identical scale effects, x must be inequality preserving. To see why suppose, by contradiction, that agent i managed to increase p_i in the new equilibrium. As all agents have identical returns to scale, this implies that i's marginal costs must have changed by a different proportion relative to some competitor. But this contradicts individual optimality, as a constituent part of equilibrium, because marginal benefits have equally changed for all agents due to $dx \neq 0$ and, by optimality, must be equated with marginal costs. Thus, either agent i or some other agent must be deviating from optimal behavior.

If the agents differ in their returns to scale, $dx \neq 0$ must induce inequality effects. To see this, consider two agents with $p_i > p_j$ and different scale effects ($\gamma_i < \gamma_j$), meaning that i's marginal costs must be less elastic than j's. Because of this property, market shares cannot remain stable: if the change in x leads to a higher common valuation $V(\cdot)$ then, as before, marginal costs of all agents must increase by the same proportion. But as marginal costs respond differentially to the market shares aspired by each agent, this necessarily implies that p_i and p_j must adjust differently.

Literature We now discuss how some of the results in Section 5.2 are related to existing literature in a narrow and a wider sense.

First, we underline that, besides identifying a unified analysis for different competition models, our inequality analysis based on "competition for market shares" allows us to tractably go beyond the standard assumption of linear variable costs in monopolistic competition. Allowing for non-CRS technologies or heterogeneous scale effects is relevant because empirical evidence shows that firm-level or sectoral heterogeneity in scale effects both exist and matter (see De Loecker et al., 2016 for a recent analysis). Our analysis highlights the implications of such heterogeneity for market inequality.

Moreover, going beyond linear costs allows us to qualify existing results. For example, our findings both complement and generalize an insight by Mrázová and Neary (2017). These authors study how certain properties of the demand function, summarized by its "demand manifold", determine the pass-through and competition effects in monopolistic competition with constant marginal cost firms. They find that, with CES demand, the level of disposable income plays no role for the dispersion of firm payoffs. Proposition 10 generalizes this result by showing that what drives the neutrality of income is not linearity, but the *premise of homogeneous scale effects*. In addition, Corollary 5 and Proposition 10 complement their finding by showing that disposable income or level variables and cost-side conditions seize to be inequality preserving once firms differ in their economies of scale, despite CES-demand. Specifically, income growth, common efficiency gains or common cost reductions lead to quantity growth jointly with increasing relative prices, a steeper price dispersion, and a growing inequality in market and payoff shares.

Second, our results provide a novel, equilibrium explanation why market inequality frequently seems to be sensitive towards changes in common market conditions. For example, to rationalize the empirically observed increasing inequality on the firm side associated with an advancing international integration, the literature has provided preference-side explanations, such as Mrázová and Neary (2017). A common implication of an advancing international integration is that it ultimately increases the amount of disposable consumer income I available to the economy, possibly through the general equilibrium effects triggered by a larger total labor force as in Melitz (2003). Because we found that an increasing disposable income, or a general upward shift in market demand, induces firm-side inequality effects whenever the firms differ in their returns to scale, our inequality analysis puts forth a *supply-side explanation*. Intuitively, market inequality increases whenever firms are differentially able to adjust to a change in a common market condition – a property, which is decisively governed by the dispersion of scale effects across firms, and not sensitive to particular forms of competition.

The general prediction that a growing income, or an increased industry-level productivity, leads to an increased market concentration fits the stylized observation that "Blockbusters", e.g., in the movie or music industry, have become more successful than ever (e.g., Elberse, 2008). If one thinks of media giants such as Disney or Sony, it is unlikely that production is subject to perfectly identical scale effects across all firms in the industry. Thus, our inequality analysis suggests that the empirical evidence about heterogeneous scale effects may be strongly connected with the one about how market inequality has evolved – a relationship which empirical literature could further seek to explore.

5.3 Idiosyncratic Valuations

We now study the case where there is ex ante agent heterogeneity in the (marginal) benefits of the aspired market shares. For simplicity, we assume that the competition for market shares verifies $p_i = t_i/T$, $T = \int t_i di$, and that the value per unit of market share does not depend on p_i, i.e., $V(i) \equiv V(i, T; x)$; we relax the latter in Section 5.4.

The key differences to the common valuation case from Section 5.2 are that i) marginal costs are not equated in equilibrium across agents, and ii) a change in a common market condition x may induce idiosyncratic effects on the equilibrium valuations of different agents. For these reasons, the inequality effects now generically depend also on properties of $V(i)$.

To make the the effects of idiosyncratic valuations most evident, we assume neutral costs, i.e., we set $\Phi(i) = c_i(p_iT)^\gamma$ by Lemma 4. Further, we suppose that x only enters $V(\cdot)$. Thus $\Pi(i) = p_iV(i, T; x) - c_i(p_iT)^\gamma$, with equilibrium condition

$$g(i) \equiv V(i, T; x) = \gamma c_i p_i^{\gamma-1} T^\gamma \equiv \varphi(i). \tag{28}$$

We take Assumption 1 as satisfied, and suppose that for any $x \in X$ the ex ante agent heterogeneity verifies $V(i, T; x) > V(j, T; x)$ iff $j \triangleright i$.[32]

[32]That is, while we generally allow for cost-side heterogeneity in c_i, this heterogeneity cannot reverse the

A first useful insight is that (28) verifies the power function structure from Section 4.4, despite heterogeneous valuations. By Proposition 7, this implies that market shares, benefit shares and payoff shares coincide. Thus, all subsequent inequality effects apply for all these market shares.

In the following, $dV_i \equiv \frac{\partial}{\partial x} V(i, T(x); x)$ denotes the impact of x on the equilibrium valuation of agent i. Our first result characterizes the existence of inequality effects under the above presumptions, and finds a simple condition assuring the occurrence of monotonic rotations.

Proposition 12 *Given equilibrium equation* (28)*, x is inequality preserving iff $V(i)$ is multiplicatively separable in i and (T, x) or, equivalently $\frac{dV(i)}{V(i)} = \frac{dV(j)}{V(j)}$, $\forall i, j$ and any $x \in X$. Further, if any $j \rhd i$ verifies*

$$\frac{dV(i)}{V(i)} > (<) \frac{dV(j)}{V(j)}, \tag{29}$$

then x induces a monotonic OR (IR) of $p(\cdot)$.

Condition (29) means that if the equilibrium valuation per unit of market share increases at a higher pace for advantaged agents, such agents will be able to increase their market (or payoff) shares as x changes. Note that, other than with common valuations, a uniform increase ($dV(i) = dV(j) > 0$) must induce a monotonic IR of $p(\cdot)$, because the valuations of advantaged agents now increase at a comparably slower pace. Further, Proposition 12 shows that the "benefit side" is neutral with respect to inequality effects iff x affects the marginal valuations by *equal proportions*. This generalizes Corollary 3 as $\frac{dV(i)}{V(i)} = \frac{dV(j)}{V(j)}$ is always satisfied for common valuations, i.e., if $V(i, T; x) = V(T; x)\ \forall i$.

The remainder of Section 5.3 is structured as follows. We study two general equilibrium examples consistent with the above analysis in Sections 5.3.1 - 5.3.2. These examples also vindicate that our approach can be used to study the inequality effects *within* a given group of agents (e.g., consumers), if different groups of agents interact with each other.[33] Section 5.3.3 briefly demonstrates that our approach to inequality also works if x is an idiosyncratic rather than a common condition.

5.3.1 Inequality Effects in an Endowment Economy

Our first example takes our inequality analysis to a general equilibrium application in a private ownership economy with a final consumption good that is competitively produced from the resources consumers sell to firms. As we shall see, the equilibrium equation determining the consumer-side dispersion of consumption (or income) is a variant of equation (28). We study the consumer-side inequality effects induced by changes in production-side fundamentals, and find that the source of the ex ante income heterogeneity is decisive for how inequality evolves. Further, we use the model to illuminate the inequality effects induced by the provision of a tax-financed unconditional basic income.

agent order implied by $V(\cdot)$.

[33]See the working paper version for a partial equilibrium model studying the effects of taxes or subsidies *within* (rather than between) the firm-side and the consumer-side, and an application to international trade, where we study the inequality effects induced by import tariffs on the domestic firms.

Model Firms and consumers are indexed by $i \in [0,1]$ and $i_c \in [0,1]$, respectively. Consumers initially own all production resources, and sell them to firms at a piece-rate $w > 0$, to which we refer, for simplicity, as a *wage rate*. Each firm uses its acquired resources y_i to produce a final consumption good with a strictly concave production function $q_i = f_i(\rho y_i)$. Formulated as a competition for market shares, firm payoff functions then are of the form,

$$\Pi_i = p_i PT - \frac{w}{\rho}\Phi(i, p_i T),$$

where P is final output price, $T \equiv \int q_i di$, $p_i \equiv \frac{Pq_i}{PT}$, and $\Phi(i,q) \equiv f_i^{-1}(q)$. For given P, T, w, optimal firm behavior then is described by the first-order condition

$$P = \tfrac{w}{\rho}\varphi(i, p_i T), \qquad \varphi(i,z) \equiv \tfrac{\partial \Phi(i,z)}{\partial z}. \tag{30}$$

Because Π_i is structurally similar to (22), the firm-side inequality effects will again be determined by the cost function. The only additional element is that obtaining $sign\, T'(x)$ is more subtle, as $T'(x)$ depends on general equilibrium forces. Nevertheless, if $\Phi(i)$ is a power function (23) with $\gamma_i = \gamma\ \forall i$ and x a consumer-side parameter, or if $T'(x) = 0$ in equilibrium, then x must always be inequality preserving on the firm-side.

Consumers spend their entire market income m_{i_c} to acquire $q^c_{i_c}$ units of the consumption good. We distinguish between two key sources of income inequality: Differences in *wage income* and differences in *capital income*. Let $\omega_{i_c} > 0$ denote the resource endowment of consumer i_c (e.g., units of effective labor), such that $\omega \equiv \int \omega_{i_c} di_c$. Given wage w, consumer i_c thus earns a wage income $w\omega_{i_c}$. Further, capital income earned by i_c is $s_{i_c}\Pi$, where s_{i_c} is an ex ante ownership share, i.e., $\int s_{i_c} di_c = 1$ and $\Pi = \int \Pi_i di$ is aggregate firm profit.[34] Both ω_{i_c} and s_{i_c} are (weakly) decreasing in consumer index i_c, such that consumers are ordered left-to-right in terms of total income m_{i_c}. Consumer i_c's consumption share $p^c_{i_c}$ coincides with her income share in this model: if $T^c \equiv \int q^c_{i_c} di_c$ denotes total demand, then

$$p^c_{i_c} \equiv \frac{Pq^c_{i_c}}{P\int q^c_{i_c} di_c} = \frac{w\omega_{i_c} + s_{i_c}\Pi}{PT^c} = \frac{m_{i_c}}{m}, \qquad m \equiv \int m_{i_c} di_c. \tag{31}$$

We recognize (31) as a variant of (28), meaning that consumer-side inequality effects are described by Proposition 12. In particular, for $V_i \equiv w\omega_{i_c} + s_{i_c}\Pi$, (29) evaluates to

$$\frac{dV(i_c)}{V(i_c)} > (<) \frac{dV(j_c)}{V(j_c)} \qquad \Longleftrightarrow \qquad \left(\frac{s_{i_c}}{s_{j_c}} - \frac{\omega_{i_c}}{\omega_{j_c}}\right)\left(\frac{d\Pi}{\Pi} - \frac{dw}{w}\right) > (<) 0. \tag{32}$$

Expression (32) shows that consumer-side inequality effects depend on the ex ante dispersions of resources and capital shares, relative to how firm profits and wages develop. In particular, consumption (and income) inequality must remain stable whenever profits and wages change at the same rate – in this case, total income of each consumer changes at the same rate, independent of how ω_{i_c} and s_{i_c} are dispersed. By contrast, if capital income is more (less) unequally dispersed than resource endowments, i.e., $s_{i_c}/s_{j_c} > (<) \omega_{i_c}/\omega_{j_c}$, then consumption

[34] For tractability, we assume that the shares of an individual consumer are equally dispersed across firms.

inequality must increase (decrease), in the sense of a monotonic OR (IR), if profits increase at a higher rate than wages.

The evolution of profits and wages is a general equilibrium outcome, which may be subtle due to the interplay between factor and final good markets. Likewise, a condition x that only enters one market side may still affect the other market side via feedback effects, making the inequality analysis not trivial. Formally, a competitive (Walrasian) equilibrium consists of two market share functions $p_i, p^c_{i_c} > 0$ and two quantities T, T^c and a price P such that p_i solves (30) for each i, $p^c_{i_c}$ is given by (31) for each i_c, and $T = T^c$. Wlog, we normalize the wage rate w to one. We now derive the emerging equilibrium inequality effects for various specific examples of condition x.

Productive Efficiency and Scale Effects For definiteness, we assume that $\Phi(i)$ is given by (23) with $\gamma_i = \gamma > 1$, $\forall i$. The two production-side parameters of interest then are productive efficiency ($x = \rho$) and returns to scale ($x = \gamma$). We have already analyzed the firm-side inequality effects of ρ and γ in a partial equilibrium context. The following proposition shows that those insights extend to the current general equilibrium setting, and complements them with consumer-side inequality effects.

Proposition 13 *Conditions ρ and γ induce the following inequality effects:*

- ***Firm-side:*** *ρ is inequality preserving, while $d\gamma > 0$ induces a monotonic IR of firm-side market (or payoff) shares.*
- ***Consumer-side:*** *ρ is inequality preserving, and $d\gamma > 0$ induces a monotonic OR (IR) of consumption shares $p^c(\cdot)$ if income inequality originates mainly from capital (resource) income, i.e., if $s_{i_c}/s_{j_c} > (<)\omega_{i_c}/\omega_{j_c}$ for each $j_c \rhd i_c$.*

The intuition is as follows. An increase in ρ stimulates overall production ($T'(\rho) > 0$), but also forces firms to lower prices, such that the real wage increases. With neutral costs, these effects level off, such that wages and profits change at the same rate, which means that $d\rho > 0$ must be inequality preserving on the consumer-side as well. Thus, while each consumer can individually afford more consumption, the dispersion of consumption and income shares is invariant to the level of productive efficiency.[35] By contrast, improved economies of scale ($d\gamma < 0$) induce firm- and consumer-side inequality effects that might go in opposite directions. E.g., $d\gamma < 0$ leads to less consumption inequality if differences in capital income are the dominant source of income inequality, while $d\gamma < 0$ simultaneously increases firm-side inequality. The reason for the former is that improved scale effects foster competition, which lowers profits more than wages.

Unconditional Basic Income Recently, the call for an "*unconditional basic income*", financed by some form of redistributive taxation, has echoed through several European coun-

[35]One might ask what happens if $d\rho > 0$ but $\Phi(i)$ is not of type (23). We can show that the inequality effects induced on the firm-side still depend exclusively on the cost function. Specifically, Corollaries 3 - 4 remain valid. Things are more subtle on the consumer side, but it can be shown that $d\rho > 0$ must always induce an OR of $p^c(\cdot)$ if $s_{i_c}/s_{j_c} > \omega_{i_c}/\omega_{j_c}$ and total consumption expenditures PT increase in ρ.

tries. We now study the inequality effects of such a policy on both market sides in the current general equilibrium context.

Section 5.2.1 established that, in a partial equilibrium context, the firm-side inequality effects induced by a quantity or sales tax τ depend exclusively on the cost function. The following shows that this result does *not* extend to the present general equilibrium context, if the tax income is uniformly redistributed to households, as mandated by the notion of an unconditional basic income. In the following proposition, we assume that τ denotes a quantity or a sales tax on final good consumption that is levied on the firm-side.

Proposition 14 *The redistributive tax τ is inequality preserving on the firm-side and leaves total production T unchanged, regardless of the cost function $\Phi(i)$, while it unambiguously reduces consumption and income inequality by inducing a monotonic IR of $p^c(\cdot)$.*

The reason why the tax must now be inequality preserving on the firm-side is that, in equilibrium, total production cannot change because τ neither affects productive efficiency nor resource endowment of this economy. But as total production remains constant, and the tax affects all firms in the sense of a level variable, the tax must be inequality preserving on the firm-side. By contrast, a (higher) basic income must induce a monotonic IR on the consumer-side, because the income of each consumer increases by the same uniform amount due to redistribution, while neither wages nor profits change due to the intervention.

It is possible to show that this result does not hinge on the assumption of a single final good sector. In fact, Proposition 14 equally applies if there is a finite number of final good sectors operated by independent firms with a unique competitive equilibrium, and the same tax is levied on all final goods sectors. However, as the next section shows, this result needs to be interpreted with caution as, in general, consumers may have a preference for consuming their resources, contradicting the perfectly inelastic supply of ω_{i_c} assumed here.

5.3.2 Inequality Effects in an Economy with a Consumption-Leisure Trade-Off

In this section, we reconsider the previous analysis by studying the case where individual labor supply is an endogenous consequence of a consumption-leisure trade-off. Specifically, we ask how the dispersions of consumption, leisure and income depends on productive efficiency, on the importance of consumption relative to leisure, and on an unconditional basic income.

Model Each consumer i_c owns one unit of perfectly divisible labor. The key difference to the previous model is that consumers value leisure, and thus face a trade-off between leisure and consumption opportunities. We follow the literature and assume a utility

$$u(i_c) = q_{i_c}^{\alpha} f_{i_c}^{1-\alpha}, \tag{33}$$

where q_{i_c} is consumption of a final good, $f_{i_c} \in [0, 1]$ is the amount of leisure, and $\alpha \in (0, 1)$ quantifies the importance of consumption relative to leisure (see, e.g., Heathcote et al., 2014). As before, $p^c_{i_c} \equiv q_{i_c}/T^c$ denotes consumer i_c's share of total consumption, where $T^c \equiv \int q_{i_c} di_c$. Besides wage income, consumer i_c receives capital income $s_{i_c}\Pi$, and possibly an unconditional basic income $\hat{\tau}$, such that total income is $m_{i_c} = (1 - f_{i_c})w + s_{i_c}\Pi + \hat{\tau}$. Thus, if P is the price of the final good and w normalized to one, the budget constraint is $p^c_{i_c}T^cP = (1 - f_{i_c}) + s_{i_c}\Pi + \hat{\tau}$. We assume that consumers are ordered left-to-right in terms of their capital shares, i.e., $s_{i_c} > s_{j_c}$ iff $j_c \rhd i_c$.

Assuming an interior solution, constrained maximization of (33) shows that $p^c_{i_c}T^cP = \alpha(1 + s_{i_c}\Pi + \hat{\tau})$ and $f_{i_c} = \frac{1-\alpha}{\alpha}p^c_{i_c}T^cP$.[36] As $m_{i_c} = \alpha(1 + s_{i_c}\Pi + \hat{\tau})$, the dispersions of consumption shares, income shares and leisure shares coincide, which is a recently rehabilitated empirical fact (Aguiar and Bils, 2015; Attanasio and Pistaferri, 2016).

Each firm i hires labor L_i to produce $q_i \geq 0$ units of the final good according to $q_i = (\frac{\rho}{c_i}L_i)^{1/\gamma}$, $\gamma > 1$. For simplicity, we focus on the case where basic income is financed by a sales tax $\tau \in [0, 1)$. Then, if p_i denotes the share of total consumption expenditure earned by firm i, its payoff is $\Pi_i = (1 - \tau)p_iPT - c_i/\rho(p_iT)^\gamma$.

A competitive equilibrium is defined identically to Section 5.3.1. It is easy to see that α, ρ, τ must all be inequality preserving on the firm-side, which simplifies the analysis, but is not decisive for consumer-side inequality effects.[37] The following proposition summarizes the consumer-side equilibrium inequality effects of labor productivity ρ, the relative importance of consumption α, and an unconditional basic income financed by tax τ.

Proposition 15 *Conditions α, ρ, τ have the following consumer-side effects*

- *Labor productivity ρ is inequality preserving in terms of consumption, leisure and income shares, but increases absolute consumption and real income gaps between different consumer types.*
- *A larger propensity to consume ($d\alpha > 0$) induces a monotonic OR of consumption, leisure and income shares, and increases aggregate production and labor supply, while real wages fall.*
- *The introduction or increase of a basic income financed by a sales tax induces a monotonic IR of consumption, leisure and income shares, but decreases labor supply and total production.*

As the effects of ρ are as before, we restrict discussion to α and τ. If consumption is more important relative to leisure ($d\alpha > 0$), consumers supply more labor to afford more consumption. The increased labor supply reduces real wages and increases profits, which benefits capital owners, and therefore increases inequality on the consumer-side. One can even show that, as real wages plunge, the poorest may end up with a *lower* consumption level, despite a higher propensity to consume.[38] The reverse prediction is that if leisure becomes more important ($d\alpha < 0$), then aggregate leisure consumption increases jointly with an increase in relative leisure f_{j_c}/f_{i_c} of the *poor*. Such a tendency has been observed, e.g., in US data (Aguiar and Hurst, 2007).[39] Further, the prediction that a higher propensity to consume leads to an increasing income and leisure inequality is reconcilable with empirical

[36]This interior solution requires that $s_{i_c}\Pi + \hat{\tau} \leq \frac{\alpha}{1-\alpha}$, which can be shown to hold in equilibrium.

[37]See the working paper version for the case of a general cost function $q_i = f_i(L_i)$.

[38]For example, if a positive mass of consumers holds no shares at all($s_{i_c} = 0$) this must always be the case.

[39]A more ambitious model could allow α to vary across consumers, e.g., because this matters empirically (Heathcote et al., 2014).

evidence comparing European countries with the US, as the latter features more income inequality jointly with a weaker preference for leisure.[40]

Introducing or increasing a basic income ($d\tau > 0$) implies a reduction in consumer-side inequality, as before. But in the current model, the equalization now comes at the costs of lower total output and consumption ($T'(\tau) < 0$). This marks a key difference to the case of inelastic labor supply, and originates from the general equilibrium property that total real production expenditures must coincide with total labor supply. Thus, as the former decreases due to the unconditional basic income in the current model, reflecting how consumers respond to the change in their income composition, total production must plunge.

Welfare Considerations The previous result is welfare relevant in the following sense. Aggregate nominal income $m = \int m_{i_c} di_c$ increases, but real income m/P decreases, as prices inflate more, due to taxation, than nominal income. For this reason, equilibrium welfare may decrease. Equilibrium utility is $u(i_c) = m_{i_c} P^{-\alpha} z(\alpha)$, $z(\alpha) \equiv \alpha^{\alpha}(1-\alpha)^{1-\alpha}$, and aggregate utility is $U = \int u(i_c) di_c = mP^{-\alpha} z(\alpha)$. It is straightforward to check that $U'(\tau) < 0$ in equilibrium. Thus, some consumers (the strongest capital owners) must necessarily be worse off due to the basic income and, depending on parameters, it may even be the case that *no* consumer gains at all. Therefore, a utilitarian planner would abstain from introducing such an unconditional basic income.

5.3.3 Idiosyncratic Conditions

We now briefly demonstrate that our approach can be applied to analyze the inequality effects if x is a purely idiosyncratic condition. Consider (28), where $c_i = c \; \forall i$, and agents are ordered according to $V(i)$, such that $p(\cdot)$ is Class I, meaning that there are finitely many different agent types. Suppose that $dV_i > 0$ only for type i, while $dV_j = 0$ for all other types. In the setting from Section 3, this corresponds to the case where x only enters $V_i(\cdot)$, such that $dV_i \equiv g_x(i, p(i), T; x) > 0$ while $dV_j \equiv g_x(j, p(j), T, x) = 0$ if j is a different type than i. A small change $dV_i > 0$ preserves the agent order as captured by $p(\cdot)$. To obtain the inequality effects caused by x, we then simply need to evaluate (8). The following result shows that all agents except i lose if $dV_i > 0$, where the loss is more pronounced the stronger the agent type is.

Proposition 16 *The idiosyncratic change $dV_i > 0$ increases p_i, while market shares p_j of all other agent types decrease proportionally.*

5.4 Market-share depending Valuations

We next study two formally more complex variants of payoff function (21) in such that the value $V(i)$ per unit of market share now depends on the level of market share $p(i)$. Such a situation arises, e.g., if the action t_i has a direct impact on $V(\cdot)$, as our first application to advertising illustrates.

[40] See, e.g., Blanchard (2004) or Maoz (2010) for leisure preferences in the US relative to Europe. Regarding income inequality, see, e.g., Federal Reserve.

5.4.1 Attention and Persuasion in Advertising: Market Concentration Effects

In this model, we study how firm-side market shares evolve depending on two key aspects of advertising: *attention-seeking* and *persuasion*. The latter captures the extent to which advertising can alter the willingness-to-pay of attentive consumers (Bagwell, 2007). Regarding the former, the competition for attention takes on the form of a *contest with endogenous prize value* $V(\cdot)$ if attention is an exhaustible and rival resource (Hefti, 2018).

Let t_i quantify firm i's advertising intensity, which affects its market share $p_i = t_i/T$, $T = \int t_i di$, due to attracting consumer attention, and possibly also the willingness-to-pay $V(\cdot)$ of attentive consumers. Specifically, we let $V_i(t_i) = \alpha t_i + \beta$ summarize how much each firm earns from its attentive consumers. The parameter $\beta > 0$ corresponds to a basic willingess-to-pay, while $\alpha \geq 0$ is the "rate of persuasion", capturing how advertising at intensity t_i converts into revenue from attentive consumers; if $\alpha = 0$, advertising only serves to attract attention. Expressed as a competition for market shares, the payoff then is

$$\Pi_i = p_i \left(\alpha p_i T + \beta\right) - \Phi(i, p_i T), \tag{34}$$

where $\Phi(\cdot)$ are advertising expenditures. Formally, (34) is a variant of (21) where, other than in previous applications, the value function $V_i(\cdot)$ now depends on p_i and T. To make the inequality effects implied by the above two aspects of advertising most evident, we assume neutral costs $\Phi(\cdot) = c_i(p_i T)^\gamma$, $\gamma \geq 2$. Further, c_i is increasing over agent types, such that $p(\cdot)$ is Class I, and we assume parameter values such that $t_i > 1$ $\forall i$ in equilibrium.[41]

The parameters α and γ determine the role of advertising for extracting consumer budget. Specifically, γ quantifies how subtle market shares respond to individual changes of advertising effort. A larger value of γ means that it is harder for the firms to influence consumer attention to their favor. More generally, high values of α, γ capture a situation, where advertising increases firm revenue mostly by persuading attentive consumers, while it is very hard for firms to increase their amount of attentive consumers. By contrast, low values of α, γ mean that a firm can best increase its revenues by competing harder for consumer attention, while the scope for increasing the willingness-to-pay of attentive consumers is small. The next result shows how these two aspects of advertising affect market concentration.

Proposition 17 *An increase in the rate of persuasion ($d\alpha > 0$) induces an OR of p_i, while less attentional control ($d\gamma > 0$) induces an IR of p_i.*

Proposition 17 offers two rationales how advertising may affect market inequality. First, *market concentration is larger* if advertising works mostly in a *persuasive manner*. Intuitively, an increase in the worthiness of attracting attention benefits the firms with the largest market shares most. Such firms thus have the strongest incentive to increase the attention they receive. Second, *market concentration must also be larger* if firms have *more influence on their chances to attract attention*. This occurs because firms with lower costs can better exploit their cost advantages, the more sensitively consumer attention responds to advertising.

[41] $\gamma \geq 2$ assures the strong quasiconcavity of $\Pi(i)$ in p_i. Further, the requirement that $t_i > 1$ $\forall i$ simplifies the proof of Proposition 17, and can be always assured, e.g. if c_i is sufficiently low or β sufficiently high.

In sum, market concentration due to advertising should be large if consumer attention and willingness-to-pay both respond sensitively towards advertising.

5.4.2 Balancing the Success Chances in Two-Prize Contests

Our last application studies the inequality effects of introducing or modifying a second prize to a contest. This is relevant, e.g., for sport economics, where assuring a certain "competitive balance" between different contestants or teams is a desired aspect of tournament design (Szymanski, 2003).[42]

Consider a contest with a prize $V_1 > 0$ for the winner, and possibly a prize $V_2 \geq 0$ for the second place, where $V_1 \geq V_2$. To focus on the inequality effects induced by multiple prizes, we assume neutral costs $\Phi(i, t_i) = c_i t_i^\gamma$, where c_i is increasing over agent types. The probability to win the first prize is $p_i = t_i/T$ while, by sampling with replacement, $(1 - p_i)p_i$ is the chance of obtaining the second prize.[43] These formulas are based on a finite number N of atomistic agents.[44] Expressed in terms of market shares, the payoff function then is

$$\Pi_i = p_i V_1 + (1 - p_i)p_i V_2 - c(i)\,(p_i T)^\gamma\,. \tag{35}$$

Formally, (35) is a variant of (21), where the value per unit of market share, $V_i(\cdot) = V_1 + (1 - 2p_i)V_2$, depends on the level of market share p_i. The following proposition shows how the prize scheme (V_1, V_2) affects the distribution of the chances p_i to seize the first prize, or to seize any prize $w_i \equiv p_i + (1 - p_i)p_i$.

Proposition 18 *V_1 is inequality preserving if $V_2 = 0$. If $\frac{dV_1}{V_1} > (<) \frac{dV_2}{V_2}$, an OR (IR) of the chance to win the first prize p_i results, which equally applies to the chance of winning any prize w_i if $p_i < 1/2\ \forall i$. If $\frac{dV_1}{V_1} = \frac{dV_2}{V_2}$, p_i is invariant to the allocation of prize money.*

Proposition 18 includes the conventional case of a single-prize contest (set $V_2 = 0$), in which case V_1 must be inequality preserving (given neutral costs), reflecting that V_1 is a level variable. This changes, however, if $V_2 > 0$, in which case neither prize, nor the prize sum $V_1 + V_2$ is a level variable, and therefore must be inequality relevant. Moreover, the two prizes have diametrically opposite effects on inequality, as an increase of the first (second) prize implies more inequality (equality) in the first-prize winning chances p_i, and likely also in the overall winning chances w_i.

With respect to designing a balanced contests, such as a sports tournament, Proposition 18 suggests that if the overall prize money $V = V_1 + V_2$ increases, then all prizes must

[42] Note that our analysis differs from the literature on contest architecture (see Konrad, 2009) as we seek to elicit the full distributive effects of a change in the prize structure, rather than identifying the reward schemes that maximize the aggregate efforts or the winning effort.

[43] Assuming sampling with replacement makes the model tractable, and at least approximates the case of sampling without replacement. By sampling with replacement we mean that if agent i did not win the first prize, for which the chance was $p_i = t_i/T$, then i competes again with the same fixed effort and the same agent pool for the second prize, i.e., the agent who won the first prize was not removed from the pool. Thus, the chance of i to win the second prize, given that i has not won the first prize, also corresponds to t_i/T.

[44] Formally, the set of agents is $I \equiv \{1/N, .., n/N, ..., 1\}$, such that $i \in [0, 1]$ for any $i \in I$, and $p_i \in [0, 1]$ is a probability mass function rather than a density. As mentioned earlier, the inequality results from Section 3 pertaining to Class I densities also apply if p_i is a probability mass function defined over a given finite set of atomistic agents.

increase proportionally ($dV_1/V1 = dV_2/V_2$) if winning odds are to remain constant. Likewise, Proposition 18 shows how changing the composition of a given prize budget $V \equiv V_1 + V_2 > 0$ affects the dispersion of success chances. Because $dV_1 < 0$ and $dV_2 > 0$ both imply an IR, ceteris paribus, it follows that an even prize split $V_1 = V_2 = V/2$ must generate the most balanced contest, while setting $V_1 = V$ (a single-prize contest) yields the most imbalanced contest. In addition, there is an effort-equality trade-off because aggregate effort $T = \int t_i di$ decreases if $V_2 \equiv V - V_1$ increases.[45]

6 Conclusion

When can changes in market conditions be exploited by certain agents to increase their equilibrium market or payoff shares, and how does the overall dispersion of these quantities change? We tried to study such questions without imposing strong assumptions on the number of ex ante different agent types, nor on the ex ante distributions of heterogeneity parameters. Our approach represents a competitive situation as a competition for market shares. We see at least two merits offered by our procedure.

First, the reformulation as a competition for market shares helps us to identify common structures in different competition models, allowing for a unified inequality analysis. For example, we find that perfect competition, monopolistic competition and homogeneous-valued contests are variations of a general form of symmetric competition. Therefore, these models deliver identical inequality predictions for comparable parameters.

In this vein, our analysis clarified that the source of the agent heterogeneity regarding the technology for "producing" market shares, such as its returns to scale, is decisive if and how equilibrium inequality changes. For example, in market competition with production the firm-side returns to scale are crucial for the inequality effects triggered by core variables, such as total income, industry-wide efficiency, or a sales tax. Such variables induce inequality effects iff firms differ in their returns to scale – a finding, which we deem particularly relevant for industries such as movie or music. It is conceivable that companies as Disney or Sony feature more favorable scale effects than smaller studios. Then, our analysis predicts that a growing consumer income, or common efficiency gains, lead to quantity growth paired with an increasing firm-side market inequality. Such patterns fit empirical evidence indicating that "Blockbuster firms" have become more successful over the last decades, and generally complements preference-based explanations for an observed increasing market inequality. Further, we showed that if all firms within a given sector have identical scale effects, but these scale effects can differ across sectors, then sectors are inequality-ranked, where those with stronger scale effects feature more market inequality, ceteris paribus. Our approach allowed to identify such predictions throughout different competition models, speaking for their robustness.

[45] To see that $T(V_1, V - V_1)$ decreases in V_1, use $V_2 = V - V_1$ in (54) and note that $\frac{\partial}{\partial V_1} p(i) > 0$. This is related to Clark and Riis (1998), who consider the case of a multi-prize contest with symmetric contestants. Their main concern is about the aggregate effort, and they find that highest aggregate effort requires to award only one prize.

The above insights appear relevant given the empirical evidence, both at the firm- and the sector-level, documenting the prevalence of heterogeneous scale effects. Moreover, our results strongly point towards a connection between the empirical evidence on heterogeneous scale effects and the one on an increasing firm-side market inequality, which to our knowledge has not yet been empirically explored.

Second, our inequality tools delivered novel predictions in the context of specific applications. The tractability of the inequality analysis spurred by our representation as a competition for market shares allowed us to swiftly derive these insights as corollaries to more abstract principles we established earlier. The application to monopolistic competition may help to make the analytical merits of our approach evident, where we were able to tractably go beyond the standard assumption of constant marginal costs. We further showed that our approach can be applied to analyze how market inequality *within* the firm- and consumer-sides depends on common market conditions in general equilibrium applications. For instance, we studied the inequality effects induced by the introduction of an “unconditional basic income”, a change in productive efficiency, or a change in the propensity to work. We thereby found that the inequality effects are governed by the same principles as in contests with idiosyncratic valuations.

Our inequality approach can be extended to other applications, some of which we indicated in text.[46] Further, this article may provide guidelines for studying normative questions related to inequality. An organizational planer may need to decide which instruments, wage schemes or prize structures to implement for obtaining a certain distributional outcome or a certain level of market concentration. Policy makers frequently are required to balance the chances of various firms for winning a grant, patent or a monopoly franchise. Likewise, sports tournament designers often care about finding a reward scheme that makes the competition most unpredictable. Such planers need to know which inequality effects can be induced by various policy instruments under the respective circumstances, which our approach can help to identify. Finally, the fact that, on most occasions, we were able to elicit the inequality effects without the need to specify the precise details of the ex ante heterogeneity can be of interest to applied work, as the precise extent of the ex ante heterogeneity may be unknown to the econometrician.

References

Acemoglu, D. and Jensen, M. K. (2010). Robust comparative statics in large static games. In *49th IEEE Conference on Decision and Control*, pages 3133–3139. IEEE.

Acemoglu, D. and Jensen, M. K. (2013). Aggregate comparative statics. *Games and Economic Behavior*, 81:27–49.

Aguiar, M. and Bils, M. (2015). Has consumption inequality mirrored income inequality? *American Economic Review*, 105(9):2725–56.

[46] The working paper version applies our approach to explore the inequality effects of firm-driven changes in product perception or the distributional impact of an import tax on domestic and foreign firms market shares. Also see Hefti and Lareida (2020) for a stand-alone application studying market consequences of competitive attention, or Hefti et al. (2020) for a dynamic application.

Aguiar, M. and Hurst, E. (2007). Measuring trends in leisure: the allocation of time over five decades. *The Quarterly Journal of Economics*, 122(3):969–1006.

Alos-Ferrer, C. and Ania, A. B. (2005). The evolutionary stability of perfectly competitive behavior. *Economic Theory*, 26:497–516.

Amegashie, J. A. (1999). The number of rent-seekers and aggregate rent-seeking expenditures: an unpleasant result. *Public Choice*, 99(1-2):57–62.

Atkinson, A. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2(3):244–263.

Attanasio, O. P. and Pistaferri, L. (2016). Consumption inequality. *Journal of Economic Perspectives*, 30(2):3–28.

Bagwell, K. (2007). The economic analysis of advertising. In Armstrong, M. and Porter, R., editors, *Handbook of Industrial Organization*, volume 3. Amsterdan: Elsevier B.V.

Blanchard, O. (2004). The economic future of europe. *Journal of Economic Perspectives*, 18(4):3–26.

Camacho, C., Kamihigashi, T., and Sağlam, Ç. (2018). Robust comparative statics for non-monotone shocks in large aggregative games. *Journal of Economic Theory*, 174:288–299.

Casella, G. and Berger, R. L. (2002). *Statistical inference*, volume 2. Duxbury Pacific Grove, CA.

Chung, T. Y. (1996). Rent-seeking contest when the prize increases with aggregate efforts. *Public Choice*, 87:55–66.

Clark, D. J. and Riis, C. (1998). Competition over more than one prize. *The American Economic Review*, 88(1):276–289.

Corchon, L. C. (1994). Comparative statics for aggregative games: The strong concavity case. *Mathematical Social Sciences*, 28(3):151–165.

Corchón, L. C. (2020). Aggregative games. *Working Paper*.

Cornes, R. and Hartley, R. (2012). Fully aggregative games. *Economics Letters*, 116(3):631–633.

De Loecker, J., Goldberg, P. K., Khandelwal, A. K., and Pavcnik, N. (2016). Prices, markups, and trade reform. *Econometrica*, 84(2):445–510.

Dixit, A. K. and Stiglitz, J. E. (1977). Monopolistic competition and optimum product diversity. *The American Economic Review*, 67:297–308.

Elberse, A. (2008). Should you invest in the long tail? *Harvard Business Review*, 86(7/8):88–96.

Heathcote, J., Storesletten, K., and Violante, G. L. (2014). Consumption and labor supply with partial insurance: An analytical framework. *American Economic Review*, 104(7):2075–2126.

Hefti, A. (2016). On the relationship between uniqueness and stability in sum-aggregative, symmetric and general differentiable games. *Mathematical Social Sciences*, 80:83–96.

Hefti, A. (2018). Limited attention, competition and welfare. *Journal of Economic Theory*, 178.

Hefti, A., Grossmann, M., and Shen, P. (2020). Heterogeneity and prize structure in dynamic contests. *Working paper*.

Hefti, A. and Lareida, J. (2020). Competitive attention, superstars and the long tail. *Working Paper*.

Jensen, M. K. (2017). Distributional comparative statics. *The Review of Economic Studies*, 85(1):581–610.

Jensen, M. K. (2018). Aggregative games. In *Handbook of Game Theory and Industrial Organization, Volume I*, pages 66–92. Edward Elgar Publishing.

Jones, J. P. (1990). Ad spending: maintaining market share. *Harvard business review*, 68(1):38–42.

Konrad, K. (2009). *Strategy and Dynamics in Contests*. Oxford University Press.

Maoz, Y. D. (2010). Labor hours in the united states and europe: The role of different leisure preferences. *Macroeconomic Dynamics*, 14(2):231.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, 71(6):1695–1725.

Morris, S. and Shin, H. S. (2002). Social value of public information. *American Economic Review*, 92(5):1521–1534.

Mrázová, M. and Neary, J. P. (2017). Not so demanding: Demand structure and firm behavior. *American Economic Review*, 107(12):3835–74.

Mrázová, M., Neary, P. J., and Parenti, M. (2016). Sales and markup dispersion: theory and empirics. *Working paper*.

Newman, M. E. (2005). Power laws, pareto distributions and zipf's law. *Contemporary physics*, 46(5):323–351.

Posner, R. A. (1992). Economic analysis of law. *Boston, Toronto and London: Little, Brown and Company*.

Szymanski, S. (2003). The economic design of sporting contests. *Journal of economic literature*, 41(4):1137–1187.

Tirole, J. (1988). *The theory of industrial organization*. MIT press.

A Proofs (for Online Publication)

Proof of Proposition 1 We first prove that $p(i) > (\geq) p(j)$ if $j > i$. Let $p > 0$ and note that $p(i) \gtreqless p \Leftrightarrow g(i,p,T) \gtreqless \varphi(i,p,T)$ because, by strong quasiconcavity (A1), $g(i,\cdot,T)$ must intersect $\varphi(i,\cdot,T)$ from above at $p(i)$ (see Figure 1 for an illustration). Further, in equilibrium

$$g(i,p(j),T) > (\geq) g(j,p(j),T) = \varphi(j,p(j),T) > (\geq) \varphi(i,p(j),T) \tag{36}$$

by Assumption 2 and (4). Hence $g(i,p(j),T) \geq \varphi(i,p(j),T)$ and thus $p(i) \geq p(j)$, where these two inequalities are strict if at least one inequality in (36) is strict. It also follows that $p(i) = p(j)$ if all inequalities in (36) are equalities, which proves the last claim of Proposition 1. Let $B(i,p,T) \equiv pV(i,p,T)$. The claims about payoffs holds because

$$\begin{aligned}\Pi(i) &= B(i,p(i),T) - \Phi(i,p(i),T) \\ &\geq B(i,p(j),T) - \Phi(i,p(j),T) > (\geq) B(j,p(j),T) - \Phi(j,p(j),T) = \Pi(j),\end{aligned}$$

where the first inequality follows from optimality and the second from Assumption 2. ■

Proof of Lemma 1 For each $i \in I$, define $z(i,p,T;x_0) \equiv \frac{g(i,p(i),T;x_0)}{\varphi(i,p(i),T);x_0)}$. Thus, using short-hand notation, (6) is $z(i) = z(j)$. Moreover, (4) implies that $z(i) = 1$ $\forall i$ in equilibrium. Total differentiation of this equation yields

$$dp(i)z_p(i) = dp(j)z_p(j) + r, \qquad r \equiv (z_T(j) - z_T(i))\,dT + (z_x(j) - z_x(i))\,dx.$$

Defining $\kappa_i \equiv z_p(i)p(i)$ and $\Delta_i \equiv \frac{dp(i)}{p(i;x_0)}$, the previous equation gives

$$\Delta_i = \Delta_j \frac{\kappa_j}{\kappa_i} + \frac{1}{\kappa_i} r. \tag{37}$$

Noting that

$$r = \left(\left(\frac{g_T(j)}{g(j)} - \frac{\varphi_T(j)}{\varphi(j)}\right) dT + \left(\frac{g_x(j)}{g(j)} - \frac{\varphi_x(j)}{\varphi(j)}\right) dx\right) - \left(\left(\frac{g_T(i)}{g(i)} - \frac{\varphi_T(i)}{\varphi(i)}\right) dT + \left(\frac{g_x(i)}{g(i)} - \frac{\varphi_x(i)}{\varphi(i)}\right) dx\right)$$

and

$$\kappa_i = z_p(i)p(i) = \frac{g_p(i)p(i)}{g(i)} - \frac{\varphi_p(i)p(i)}{\varphi(i)} = \varepsilon_i - \eta_i,$$

(37) becomes

$$\Delta_i = \Delta_j \frac{\eta_j - \varepsilon_j}{\eta_i - \varepsilon_i} + \frac{1}{\eta_i - \varepsilon_i}(-r) = \Delta_j k_{ij} + R_{ij}$$

which yields (8). The claims that $\eta_i > 0$ and $\varepsilon_i < \eta_i$ follow from strong quasiconcavity (A1) in Assumption 1. ■

Proof of Theorem 1 We prove the first claim by contradiction. Hence suppose that $R = 0$ $\forall i, j \in [0, 1]$ and any $x \in X$, but $\exists j \in (0, 1)$ such that $\Delta_j \neq 0$ (equivalently $dp(j) \neq 0$). Because in equilibrium the condition

$$\int \frac{\partial p(s; x)}{\partial x} ds = 0 \tag{38}$$

must hold, we can then suppose, wlog, that $\Delta_j > 0$ for some $j \in (0, 1)$. By (8) we must have $\Delta_i > 0$ for all $i < j$, and because of (38) $\exists j' \in (0, 1)$, $j' > j$, such that $\Delta_i < 0$ for all $i > j'$. Take $i < j$ and $i' > j'$. Then $\Delta_i > 0$ but also $\Delta_i = k_{ii'} \Delta_{i'} < 0$, contradiction. Turning to the second claim, note that if $R \neq 0$ for some i, j then $\Delta_i = 0$ $\forall i \in [0, 1]$ is impossible by (8). Hence $\forall x \in X$ $\exists i$: $\Delta_i(x) \neq 0$, or equivalently $\frac{\partial p(i;x)}{\partial x} \neq 0$, and therefore $\exists \delta > 0$ such that $p(i; x') \neq p(i; x)$ for $x' \in (x - \delta, x + \delta)$. ■

Proof of Theorem 2 Step 1: We prove the first claim, restricting attention to the OR-case (the IR-case is similar). Because $R(x_0)$ is uniformly positive, $\exists i \in (0, 1)$: $\Delta_i(x_0) > 0$ by the proof of Theorem 1. By the integral condition (38), there then must also be $i' \in (0, 1)$: $\Delta_{i'}(x_0) < 0$. It then follows from (8) that $i_0 = \sup\{i \in [0, 1] : \Delta_i(x_0) > 0\} \in (0, 1)$, $i_1 = \inf\{i \in [0, 1] : \Delta_i(x_0) < 0\} \in (0, 1)$ and $i_0 \leq i_1$. For any $i < i_0$: $\Delta_i(x_0) > 0$ and hence $\frac{\partial p(i;x_0)}{\partial x} > 0$. This derivative condition implies that $\forall i < i_0$ $\exists \delta_i > 0$: $p(i; x) > p(i; x_0)$ $\forall x \in (x_0, x_0 + \delta_i)$.

Step 2: Because $p(\cdot; x)$ is Class I, there is a finite number of equivalence classes to the left of i_0, and we only need to consider a single i, with corresponding δ_i, for each step of $p(\cdot; x)$ to the left of i_0. Let $\delta^0 > 0$ be the smallest value of these δ_i. We have thus shown that $\exists$ $i_0 \in (0, 1)$ such that for any given $x \in (x_0, x_0 + \delta^0)$ we have $p(i; x) > p(i; x_0)$ for $i < i_0$. A similar argument shows that we can find $\delta^1 > 0$ such that $\exists$ $i_1 \in (0, 1)$ such that $p(i; x) < p(i; x_0)$ for $i > i_1$ and any $x \in (x_0, x_0 + \delta^1)$. Let $\delta \equiv \min\{\delta^0, \delta^1\} > 0$. Summarizing, the arguments so far show that $\exists i_0, i_1 \in (0, 1)$, $i_0 \leq i_1$ such that for $x \in (x_0, x_0 + \delta)$ we have $p(i; x) > p(i; x_0)$ for $i < i_0$ and $p(i; x) < p(i; x_0)$ for $i > i_1$. If $\Delta_i \neq 0$ for any $i \in (i_0, i_1]$ we must have $i_0 = i_1$ and the proof is complete. Now suppose that $\exists m \in (i_0, i_1]$: $\Delta_m(x_0) = 0$. Then (8) implies that $\Delta_i > 0$ for any $m \rhd i$, and $\Delta_j < 0$ for any $j \rhd m$. But this shows that there can be at most one step of $p(\cdot; x)$ for which $\Delta_m(x_0) = 0$. It follows that independent of whether $p(m; x') >=< p(m; x)$ for $x \in (x_0, x_0 + \delta)$, $p(\cdot; x')$ must be OR of $p(\cdot; x_0)$.

We now prove the second claim. By step 1 and the global uniform positivity of R, we must have $\Delta_0(x) > 0$ and thus $\frac{\partial p(0;x)}{\partial x} > 0$ for any $x > x_0$ (note that this result is valid also if $p(\cdot)$ is of Class II), hence $p(i; x') > p(i; x_0)$ $\forall i \in [0]$. Similarly, $\Delta_1(x) < 0$ for all $x > x_0$, hence $p(i; x') < p(i; x_0)\ \forall i \in [1]$. ■

Proof Lemma 2 $u'v \geq uv'$ hence also $u'v - uv \geq uv' - uv$ or $u' - u \geq \frac{u}{v}(v' - v) > v' - v$. ■

Proof of Theorem 3 We only prove the OR case, and show that (12) implies condition (11). Noting the equivalence between (12') and (12), we define $f(x; i, j) \equiv \frac{p(i;x)}{p(j;x)}$. If $p(\cdot)$ is Class II and (12') is satisfied, then $f(x; i, j) > f(x_0; i, j)$ whenever $x > x_0$, and the claim follows from the fact that condition (11) induces a monotonic rotation. If $p(\cdot)$ is Class I, then $p(\cdot; x)$ is piecewise constant for any given $x \in X$, with a finite number of downward jumps. If (12') is satisfied for any two $i, j \in (0, 1)$ with $j \rhd i$ that are not discontinuities of $p(\cdot; x)$, then $f(x; i, j) > f(x_0; i, j)$ follows for any such i, j and any $x > x_0$, proving the claim also for Class I densities. ■

Proof of Corollary 1 Define $dp(i; x) \equiv \frac{\partial p(i;x)}{\partial x}$. If x is inequality preserving, then we must have $dp(i; x) = 0\ \forall i \in I$ and $\forall x \in X$. But as

$$\frac{\partial}{\partial x}\left(\frac{p(i;x)}{p(j;x)}\right) = 0 \quad \Leftrightarrow \quad \frac{dp(i;x)}{p(i;x)} = \frac{dp(j;x)}{p(j;x)} \tag{39}$$

we immediately obtain that $\frac{\partial}{\partial x}\left(\frac{p(i;x)}{p(j;x)}\right) = 0$, $\forall i, j \in I$ and $\forall x \in X$ whenever x is inequality preserving. In the other direction, suppose that $\frac{\partial}{\partial x}\left(\frac{p(i;x)}{p(j;x)}\right) = 0\ \forall i, j \in I$ and $\forall x \in X$ applies. Then, (39) implies that $\frac{dp(i;x)}{p(i;x)} = k(x) > 0\ \forall i \in I$, or equivalently $dp(i; x) = k(x)p(i; x)$. Because $p(\cdot; x)$ is a density, it follows that $\int_I dp(s; x)ds = 0$ as well as $\int_I p(s; x)ds = 1$. Integrating $dp(i; x) = k(x)p(i; x)$ on both sides delivers that $k(x) = 0$, which assures that $dp(i; x) = 0\ \forall i \in I$ and any $x \in X$, meaning that x must be inequality preserving. ■

Proof of Proposition 2 We only prove the first claim as the remaining claims are proved identically. Recall from the equivalence class argument in step 2 of the proof of Theorem 2 that, because R is uniformly positive, there is a finite number of leading agents with $\Delta_i(x_0) > 0$, possibly a single $\Delta_m(x_0) = 0$ and a finite number of weaker agents with $\Delta_j(x_0) < 0$. Define $f(i, j, x) = \frac{p(i,x)}{p(j,x)}$. If $\Delta_{i'}(x_0) \geq 0$ then any i with $i' \rhd i$ has $\Delta_i(x_0) > \Delta_{i'}(x_0)$ by (8) and the fact that $k_{ij} \geq 1$. Hence we must have $\frac{\partial f(i,i',x_0)}{\partial x} > 0$. If $\Delta_{i'}(x_0) < 0$ but $\Delta_i(x_0) > 0$, then obviously $\frac{\partial f(i,i',x_0)}{\partial x} > 0$ Thus for any pair (i, i') as described above $\exists \delta_{i,i'} > 0$ such that $f(i, i', x') > f(i, i', x_0)$ for all $x' \in (x_0, x_0 + \delta_{i,i'})$. The proof is completed by letting $\delta > 0$ be the smallest among these (finitely many) $\delta_{i,i'}$ and δ^0, δ^1 as in the proof of Theorem 2. ■

Proof of Theorem 4 The claim follows from Theorem 3 because, by (8), if $R(x)$ is globally uniformly positive (negative) and $k = 1$, then $\Delta_i(x) > (<) \Delta_j(x)$, for any $j \rhd i$ and any $x \in X$, and hence condition (12) holds. ■

Proof Theorem 5 Define $g_x(i) \equiv \frac{\partial g(i,p(i),T(x);x)}{\partial x}$ and $\varphi_x(i) \equiv \frac{\partial \varphi(i,p(i),T(x);x)}{\partial x}$. Then, by (8), $A(i) = \frac{\partial_x g(i)}{g(i)} - \frac{\partial_x \varphi(i)}{\varphi(i)}$.

(If). Suppose that $\frac{g(i)}{\varphi(i)} = u(i, p(i))H(T(x); x)$ $\forall x \in X$. Then $Ln\,(g(i)) - Ln\,(\varphi(i)) = Ln\,(u(i, p(i))) + Ln\,(H(T(x); x))$ and differentiation wrt x shows that $A(i) = \frac{H_T T'(x) + H_x}{H(T(x);x)}$, which is independent of $(i, p(i))$. Thus $A(i) - A(j) = 0$, or equivalently $R_{ij} = 0$ $\forall i, j$, showing that x is inequality preserving by Theorem 1.

(Only if). Suppose that $R_{ij} = 0$ $\forall i, j$ and any $x \in X$. Then equivalently, $A(i) = A(j)$, $\forall i, j$ and any $x \in X$, which implies that $A(i)$ must be a function that does not depend on $(i, p(i))$, i.e., $A(i) = h(T(x); x)$ in general. Thus $\frac{g_x(i)}{g(i)} - \frac{\varphi_x(i)}{\varphi(i)} = h(T(x); x)$ $\forall x \in X$, and integration wrt x implies that $\frac{g(i)}{\varphi(i)}$ is generally of the form $\frac{g(i)}{\varphi(i)} = u(i, p(i))e^{H(T(x);x)}$, where $H(T(x); x)$ is the anti-derivative of $h(T(x); x)$. ■

Proof of Lemma 4 If $\Phi(i, t) = c(i)t^\gamma$, then $\varphi(i, p(i), T) = \gamma c(i)p(i)^{\gamma-1}T^\gamma$ in the transformed model, showing the multiplicative separability of $\varphi(i)$ in (i, p) and T. For the converse, note that

$$\varphi(i) \equiv \frac{\partial \Phi(i, p(i)T)}{\partial p(i)} = h(i, p(i)T)T,$$

where $h(i, p(i)T) \equiv \Phi_t(i, p(i)T)$. Then

$$\frac{\varphi_T(i)}{\varphi(i)} = \frac{h_t(i, p(i)T)p(i)}{h(i, p(i)T)} + \frac{1}{T}, \tag{40}$$

where h_t denotes the partial derivative with respect to the 2nd argument of $h(i, \cdot)$. If costs are neutral, then $\frac{\varphi_T(i)}{\varphi(i)} = \frac{\varphi_T(j)}{\varphi(j)}$ $\forall i, j$ and any $p(i), T > 0$ or, equivalently, $\frac{h_t(i,t(i))t(i)}{h(i,t(i))} = \frac{h_t(j,t(j))t(j)}{h(j,t(j))}$ $\forall i, j$ and any $t(i) > 0$, by (40), which implies that $\frac{h_t(i,t(i))t(i)}{h(i,t(i))} = \chi$ (a constant) $\forall i, t(i)$. This implies, by integration wrt $t(i)$, that necessarily $h(i, t(i)) = w(i)t(i)^\chi$, and the fact that $\int h(i, t)dt = \Phi(i, t)$ gives $\Phi(i, t(i)) = \frac{w(i)}{\chi+1} t(i)^{\chi+1}$. ■

Proof of Proposition 3 The first claim directly follows from Theorem 5. For the converse, note that $R_{ij} = 0$ $\forall x \in X$ by Theorem 1, because x is inequality preserving. Thus, as $T'(x) \neq 0$, this shows that $A(i) = A(j)$ $\forall i, j$, which in turn implies that $\frac{\varphi_T(i,p,T)}{\varphi(i,p,T)} = h(T)$

$\forall i, T$. Integrating with respect to T shows the multiplicative separability of $\varphi(i)$ in (i, p) and T. ■

Proof of Proposition 4 In text. ■

Proof of Corollary 2 Let $j \rhd i$. The power function property implies that $\varepsilon(i) = \varepsilon(j) = \alpha$ in (7). Then, $k_{ij} = \frac{\eta_j - \alpha}{\eta_i - \alpha}$, and the fact that $p(i) = t(i)/T$ further implies that $\psi(i) = \eta_i \ \forall i$. Thus, as $T'(x) > 0$ we have $R_{ij} > (<) 0 \Leftrightarrow \psi(j) > (<) \psi(i) \Leftrightarrow k_{ij} > (<) 1$. This shows that x induces an OR (IR) if $\psi(j) > (<) \psi(i)$, and the respective rotation must be partially monotonic by Proposition 2. ■

Proof of Proposition 5 Recall from (8) that $k_{ij} \equiv \frac{\eta_j - \varepsilon_j}{\eta_i - \varepsilon_i}$, where $\eta_i \equiv \frac{\varphi_p(i)p(i)}{\varphi(i)}$ and $\varepsilon_i \equiv \frac{g_p(i)p(i)}{g(i)}$ for any i. Suppose that (17) holds. Differentiation and the fact that $g(i) = \varphi(i)$ in equilibrium yields

$$\frac{\varphi_p(i)p(i)}{\varphi(i)} - \frac{g_p(i)p(i)}{g(i)} = \xi(T; x)$$

in equilibrium $\forall i$, which shows that $\eta_i - \varepsilon_i = \xi(T; x) \ \forall i$. This assures that $\eta_j - \varepsilon_j = \eta_i - \varepsilon_i$ $\forall i, j$. Thus $k_{ij} = 1$, and the claim about monotonic rotations follows from Theorem 4. For the converse, fix any $T > 0$ and any x, and let $\eta_j - \varepsilon_j = \eta_i - \varepsilon_i$ for any given i, j. This implies that $\eta_i - \varepsilon_i$ must be entirely independent of the agent index $\forall i$, i.e.,

$$\eta_i - \varepsilon_i = \frac{\varphi_p(i)p(i)}{\varphi(i)} - \frac{g_p(i)p(i)}{g(i)} \equiv c(T; x)$$

for any i and any given $p > 0$. Integration with respect to p then shows that

$$\frac{\varphi(i, p, T; x)}{g(i, p, T; x)} = z(i, T; x) p^{c(T;x)},$$

which corresponds to (17). ■

Proof of Proposition 6 If $\Pi(i) = U(i, p(i))v(T; x) - Y(i, p(i))z(T; x)$, then

$$g(i) \equiv \frac{\partial U(i, p(i))v(T; x)}{\partial p(i)} = u(i, p(i))v(T; x),$$

where $u(i, p(i)) \equiv \frac{\partial U(i, p(i))}{\partial p(i)}$. This shows that $g(i)$ is multiplicatively separable in $(i, p(i))$ and (T, x). As the same holds for $\varphi(i)$, the ratio $\frac{g(i)}{\varphi(i)}$ must also be multiplicatively separable, and therefore $p(\cdot)$ must be invariant to x by Theorem 5. Consider next benefit shares $b(\cdot)$. Corollary 1 implies that $b(\cdot)$ is invariant to x iff the ratio $\frac{b(i;x)}{b(j;x)}$ does not depend on x for any

i, j. Multiplicative separability assures that this ratio is of the form $\frac{b(i;x)}{b(j;x)} = \frac{z(i,p(i))}{z(j,p(j))}$, which must be invariant to x as $p(\cdot)$ is invariant to x. The claim about $e(\cdot)$ is proved identically. ■

Proof of Lemma 5 For the first claim, to see the "if" (the "only if" is obvious), suppose by contradiction that, wlog, there is i for which $q(i) > r(i)$. But because $\frac{q(i)}{r(i)} = \frac{q(j)}{r(j)}$ $\forall j \neq i$, it follows that also $q(j) > r(j)$ $\forall j \neq i$, which is impossible as both densities must integrate to one. For the remaining claim, suppose that $\frac{q(i)}{q(j)} > \frac{r(i)}{r(j)}$ $\forall j \rhd i$. Because $q(\cdot)$ and $r(\cdot)$ both are SSD densities with the same equivalence classes $[i]$, it follows from Definition 9 that $q(\cdot)$ must be a monotonic OR of $r(\cdot)$ (and similar for the IR case). ■

Proof of Proposition 7 Follows from Lemma 5 and the fact that $\frac{s(i)}{s(j)} = \frac{b(i)}{b(j)} = \frac{e(i)}{e(j)}$. ■

Proof of Proposition 8 1) follows from Lemma 5 as $\frac{p_i}{p_j} = \frac{b_i}{b_j}$, $\forall i, j$. 2) Proposition 7 shows that $s(i) = b(i) = e(i)$ $\forall i$. Moreover, note that (27) is a version of (18) with $\alpha = 1$ and $\hat{g}(i, T; x) \equiv V(T; x)$. Thus (20) shows that $\frac{s_i}{s_j} = \frac{p_i}{p_j}$, and $p_i = s_i$ $\forall i$ follows from Lemma 5. ■

Proof of Corollary 3 If x only affects $V(\cdot)$, then x must be a level variable, and the first claim follows from Proposition 3, noting that $T'(x) \neq 0$, $\forall x \in X$. Let x be a cost-side condition, and thus also $T'(x) > 0$ $\forall x \in X$ as noted in the main text. Multiplicative separability means that the function $\varphi(\cdot)$ verifies $\varphi(i) = u(i, p_i)w(T; x)$. Thus, the equilibrium equation (4) $g(i) = \varphi(i)$ can be equivalently reformulated as

$$\hat{g}(i; x) \equiv \frac{V(T)}{w(T; x)} = u(i, p_i) \equiv \hat{\varphi}(i)$$

iff $\varphi(i)$ is multiplicatively separable, which implies that x satisfies the definition of a level variable in the new equilibrium system. Thus, the second claim follows from the first. ■

Proof of Corollary 4 As $T'(x) > 0$ $\forall x \in X$, the claim follows from Proposition 4. ■

Proof of Corollary 5 The first claim directly follows from Corollary 3 noting that $\varphi(i)$ is multiplicatively separable in (i, p) and (T, x) iff $\gamma_i = \gamma$ $\forall i$. For the second claim, suppose that equilibrium market shares $p(\cdot)$ satisfy $\gamma_i < \gamma_j \Leftrightarrow p_i > p_j \Leftrightarrow j \rhd i$, $\forall x \in X$. By presumption, marginal costs are of the form $\varphi(i) = h(i)p_i^{z(\gamma_i)}w(i, T)u(x)$, where either $z(\gamma_i) = \frac{\gamma_i \eta}{\eta - 1} - 1$ for monopolistic competition, and else $z(\gamma_i) = \gamma_i - 1$ by (23). In the latter case, if x only enters $V(\cdot)$ (hence $u(x) = 1$), i.e., is a level variable, the claim follows from Corollary 2 as $T'(x) > 0$

(as $V_x > 0$), $p_i = t_i/T$ and $\psi_i = \gamma_i - 1$, and thus $\psi_i < \psi_j$ $\forall j \rhd i$. A similar reformulation as in the proof of Corollary 4 (dividing equation (4) by $u(x)$) shows the claim if x is a cost-side condition instead.

In case of monopolistic competition, (4) yields

$$g(i) \equiv I = w_i I^{\gamma_i} \frac{\gamma_i \eta}{\eta - 1} p_i^{\frac{\gamma_i \eta}{\eta-1} - 1} T^{\frac{\gamma_i}{\eta-1}} \equiv \varphi(i), \qquad w_i \equiv \frac{c_i}{\rho} r_i^{-\frac{\gamma_i \eta}{\eta-1}}. \tag{41}$$

Note that $x = \rho$ is the only parameter that enters the cost in a multiplicative separable way as required by Corollary 5 given that $\gamma_i \neq \gamma_j$. Evaluating (9) for (41) and $x = \rho$ gives

$$sign(R_{ij}) = sign\left(A(i) - A(j)\right) = sign\left((\gamma_j - \gamma_i)\frac{T'(\rho)}{T}\right), \quad j \rhd i, \tag{42}$$

where $T'(\rho) > 0$. Because $\gamma_j > \gamma_i$, this assures that R_{ij} is uniformly positive, which implies that $d\rho > 0$ induces an OR of $p(\cdot)$ by Theorem 2. To see that this OR must be partially monotonic, note from (8) and (41) that $k_{ij} = \frac{\eta_j}{\eta_i}$, where $\eta_s = \frac{\gamma_s \eta}{\eta - 1} - 1$, which further implies that $k_{ij} > 1$ uniformly $\forall j \rhd i$ as $\gamma_i < \gamma_j$ $\forall j \rhd i$. This exactly amounts to the condition of Proposition 2 that assures a partially monotonic rotation in the sense of (13). ■

Proof of Corollary 6 Let $j \rhd i$. If $\varphi(i) = \gamma c_i p_i^{\gamma-1} T^{\gamma}$, then

$$\frac{p_i}{p_j} = \left(\frac{c_j}{c_i}\right)^{\frac{1}{\gamma-1}}.$$

As $j \rhd i$ the ratio $\frac{p_i}{p_j}$ must strictly decrease in γ, and the claim follows from Theorem 3. In case of monopolistic competition, (41) shows that

$$\frac{p_i}{p_j} = \left(\frac{c_j}{c_i}\right)^{\frac{\eta-1}{\eta(\gamma-1)+1}} \left(\frac{r_i}{r_j}\right)^{\frac{\gamma\eta}{\eta(\gamma-1)+1}}, \tag{43}$$

which also decreases strictly in γ if $j \rhd i$, and the claim also follows from Theorem 3. ■

Proof of Proposition 9 The claim follows from (43) and Theorem 3, because $\frac{p_i}{p_j}$ is strictly increasing in η whenever $j \rhd i$. ■

Proof of Proposition 10 Note that the optimality condition (41) can be stated as

$$I^{1-\gamma_i} = w_i \frac{\gamma_i \eta}{\eta - 1} p_i^{\frac{\gamma_i \eta}{\eta-1} - 1} T^{\frac{\gamma_i}{\eta-1}} \tag{44}$$

The procedure from Section 3.4 can be used to verify that $T'(I) < 0$. Applying (15) to this equation directly shows that $A(i) - A(j) = 0 \; \forall i, j$ iff $\gamma_i = \gamma \; \forall i$, which proves the first claim by Theorem 1. For the remaining claim, let $j \rhd i$ and the ex ante heterogeneitsy be such that $p_j < p_i$ whenever $\gamma_j > \gamma_i$. Further, plugging the fact that

$$q_i = I p_i^{\frac{\eta}{\eta-1}} T^{\frac{1}{\eta-1}} \tag{45}$$

into (41) and rearranging yields

$$\frac{\eta - 1}{\eta} \left(\frac{I}{T} \right)^{\frac{1}{\eta}} = \frac{\gamma_i}{\rho} q_i^{(\gamma_i - 1) + \frac{1}{\eta}}, \tag{46}$$

and thus

$$\frac{q_i^{(\gamma_i - 1) + \frac{1}{\eta}}}{q_j^{(\gamma_j - 1) + \frac{1}{\eta}}} = \frac{\gamma_j}{\gamma_i}. \tag{47}$$

Now, because $T'(I) < 0$, (46) implies that $dq_i > 0 \; \forall i$. As $dq_i > 0 \; \forall i$ and $\gamma_j > \gamma_i$, (47) implies that the ratio $\frac{q_i}{q_j}$ must also increase strictly if $dI > 0$. By (45), $\frac{q_i}{q_j} = \left(\frac{p_i}{p_j} \right)^{\frac{\eta}{\eta-1}}$, which shows that the ratio $\frac{p_i}{p_j}$ increases in I by the previous step. This proves the monotonic OR of $p(\cdot)$ by Theorem 3. ■

Proof of Proposition 11 From $p_i = r^\eta P_i^{1-\eta}/T$ we first observe that P_i is increasing over firm types, and second that $\frac{P_i}{P_j} = \left(\frac{p_j}{p_i} \right)^{\frac{1}{\eta-1}}$. Thus, the first claim follows from Corollary 1. For the second claim, let $j \rhd i$, and note that the ratio $\frac{p_i}{p_j}$ must strictly increase in I and ρ by the proof of Proposition 10. As $\pi(\cdot)$ is increasing, the fact that $\frac{P_i}{P_j}$ is decreasing in I and ρ implies that $\pi(\cdot)$ must rotate (monotonically) counter-clockwise, by the same logic that an increasing ratio $\frac{p_i}{p_j}$ causes a monotonic OR of $p(\cdot)$ (i.e, a clockwise rotation of $p(\cdot)$). ■

Proof Proposition 12 As costs are neutral, (15) and the definition of $dV(i)$ imply that

$$sign\, R_{ij} = sign\; (A(i) - A(j)) = sign \left(\frac{dV(i)}{V(i)} - \frac{dV(j)}{V(j)} \right), \tag{48}$$

and the first claim follows from Theorem 1. For the second claim, note from (28) that the function $\frac{\varphi(i)}{g(i)}$ verifies

$$\frac{\varphi(i)}{g(i)} = \frac{\gamma c_i T^\gamma}{V(i, T; x)} p_i^{\gamma - 1},$$

which satisfies (17). Thus, by Proposition 5, $dx > 0$ induces a monotonic OR (IR) if $R_{ij} > (<) 0$, $\forall j \rhd i$. The second claim follows as (48) implies that $R_{ij} > (<) 0$ for any $j \rhd i$

iff (29) holds. ■

Proof Proposition 13 Regarding firms, optimality condition (30) and $w = 1$ show that $\rho P = \gamma c_i p_i^{\gamma-1} T^{\gamma-1}$. This equation shows that ρ must be inequality preserving (due to multiplicative separability) while $d\gamma > 0$ must induce a monotonic IR of $p(\cdot)$, independent of any general equilibrium effects. Regarding consumers, the normalization $w = 1$ requires us to set $dw = 0$ in (32). Further, equation (31) and $w = 1$ imply that $PT = \omega + \Pi$ in equilibrium, which together with the fact that $\Pi = PT\frac{\gamma-1}{\gamma}$ yields $\Pi = (\gamma - 1)\omega$. Thus $\Pi'(\rho) = 0$, which implies that $dV(i_c)/V(i_c) = dV(j_c)V(j_c)$ for any i_c, j_c, which shows that ρ must be inequality preserving by Proposition 12. By contrast, the fact that $\Pi'(\gamma) > 0$ implies that the sign of $s_{i_c}/s_{j_c} - \omega_{i_c}/\omega_{j_c}$ becomes decisive for the inequality effects induced by γ, and the remaining claim formally follows again from Proposition 12. ■

Proof Proposition 14 Wlog, we normalize $\rho = 1$. Consider a quantity tax, such that τT is total tax income. Then, $w = 1$ and integrating (31) shows that $PT = \omega + \Pi + \tau T$. Together with $\Pi = \int \Pi_i di = (P - \tau)T - \int \Phi(i, p_i T)di$, this implies that $\omega = \int \Phi(i, p_i T)di$. The last equation, in turn, implies that $T'(\tau) = 0$ in equilibrium. To see this, note from this equation that $0 = \int \varphi(i, p_i T) \left(p_i'(\tau)T + p_i T'(\tau)\right) di$, where by (30) $\varphi(i, p_i T) = P - \tau \ \forall i$ in equilibrium. The claim follows by noting that always $\int p_i'(\tau)di = 0$ and $\int p_i di = 1$. Then, the fact that $T'(\tau) = 0$ implies, by (30), that τ must be inequality preserving on the firm-side. This further implies, again by (30), that $P'(\tau) = 1$, which in turn shows that $\Pi'(\tau) = 0$. Individual consumer income is $\omega_{i_c} + s_{i_c}\Pi + \tau T$. Thus, by the logic of (32), $d\tau > 0$ induces a monotonic IR of $p^c(\cdot)$ if $(\omega_{i_c} - \omega_{j_c}) + (s_{i_c} - s_{j_c})\Pi > 0$, $\forall j_c \rhd i_c$, which holds by presumption. If τ is a sales tax, such that $(1 - \tau)PT$ is total tax income, repeating exactly the same steps as above completes the proof. ■

Proof of Proposition 15 Note that α, ρ, τ are inequality preserving for firms. Equilibrium profits are $\Pi_i = (1-\tau)p_i PT\frac{\gamma-1}{\gamma}$, and thus aggregate profits are $\Pi \equiv \int \Pi_i di = (1-\tau)PT\frac{\gamma-1}{\gamma}$. The equilibrium equation determining consumer market shares ($T^c = T$) is

$$\alpha(1 + s_{i_c}\Pi + \tau PT) = p_{i_c}^d TP, \tag{49}$$

hence $PT = \alpha(1 + \Pi + \tau PT)$ by aggregation. Substituting the expression for Π yields

$$\Pi = \frac{\alpha(1-\tau)(\gamma-1)}{\gamma(1-\alpha) + \alpha(1-\tau)}, \qquad PT = \frac{\alpha\gamma}{\gamma(1-\alpha) + \alpha(1-\tau)}. \tag{50}$$

Further, the firm-side optimality condition is $(1-\tau)PT = c_i\gamma p_i^{\gamma-1}T^\gamma/\rho$, which implies that

$$T^\gamma = \frac{(1-\tau)\rho}{K}\frac{\alpha}{\gamma(1-\alpha)+\alpha(1-\tau)}, \qquad K \equiv \int c_i p_i^{\gamma-1} di. \tag{51}$$

Equation (49) is of form (28), such that a monotonic OR (IR) of consumption shares $p_{i_c}^c$ (or leisure or income shares) occurs if

$$(s_{i_c} - s_{j_c})\left(d\Pi + \hat{\tau} d\Pi - \Pi d\hat{\tau}\right) > (<) 0, \qquad \hat{\tau} \equiv \tau PT, \tag{52}$$

whenever $j_c \rhd i_c$ by Proposition 12.

Regarding $d\rho > 0$, (51) shows that $\Pi'(\rho) = \hat{\tau}'(\rho) = 0$, which by (52) implies that ρ must be inequality preserving on the consumer-side. As also $T'(\rho) > 0$ by (51) and $q_{i_c} = p_{i_c}^c T$, $dq_{i_c} > 0$ for each consumer follows, which implies that absolute consumption gaps $q_{i_c} - q_{j_c}$, $j_c \rhd i_c$, must increase by Lemma 2 as q_{i_c}/q_{j_c} remains constant. Nominal income $m_{i_c} = \alpha(1+s_{i_c}\Pi+\hat{\tau})$ remains constant, but real income m_{i_c}/P increases as $P'(\rho) < 0$ by (50) and (51). As (real) income shares remain constant, it again follows that the absolute gaps of real income must increase.

Regarding $d\alpha > 0$, (50) shows that $\Pi'(\alpha), \hat{\tau}'(\alpha) > 0$. Let $j_c \rhd i_c$. For (52) to hold, we thus require that $\frac{\Pi'(\alpha)}{\Pi} > \frac{\hat{\tau}'(\alpha)}{1+\hat{\tau}}$, which by (50) is equivalent to $\frac{\gamma}{\gamma-1} > \alpha(1-\tau)$. As the last inequality is satisfied, we conclude that $d\alpha > 0$ induces a monotonic OR of p_{i_c}. Further, larger production $T'(\alpha) > 0$ follows from (51), which necessarily requires an increase in total labor supply. Finally, (50) and (51) imply that $P'(\alpha) > 0$, which implies that real wages $(= 1/P)$ must fall.

Regarding $d\tau > 0$, (50) shows that $\Pi'(\tau) < 0$ but $\hat{\tau}'(\tau) > 0$, which by (52) implies an IR of $p_{i_c}^c$. Further, (51) shows that $T'(\tau) < 0$; hence total production and total labor supply decreases. ■

Proof of Proposition 16 We prove the claim by applying Lemma 1. First, we note that $k_{mn} = 1\ \forall m, n \in [0,1]$. Second, we note that $R_{mn} = 0$ whenever $m, n \neq i$, and $R_{ij} = \frac{dV_i}{V_i} > 0$ whenever $j \rhd i$, while $R_{ji} = -\frac{dV_i}{V_i} < 0$ whenever $i \rhd j$. These facts imply that $\Delta_m = \Delta_n$ whenever $m, n \neq i$. We now prove that $\Delta_i > 0$ and $\Delta_j < 0\ \forall j \neq i$. By contradiction, suppose that $\Delta_j \geq 0$ for some $j \neq i$. Then actually $\Delta_j > 0$ for all $j \neq i$. Because market shares must integrate to unity, it follows that $\Delta_i < 0$, which is impossible because $\Delta_i = \Delta_j + R_{ij} > 0$. We conclude that $\Delta_i > 0$ while $\Delta_j < 0\ \forall j \neq i$, which further implies that only the market share of i increases. The claimed proportionality in the decreasing market shares of the other agents follows directly from $\Delta_m = \Delta_n$, $m, n \neq i$. ■

Proof of Proposition 17 The equilibrium equation is

$$g(i) \equiv 2\alpha p_i T + \beta = \gamma c_i p_i^{\gamma-1} T^{\gamma} \equiv \varphi(i). \tag{53}$$

We first use the procedure in Section 3.4 to determine the sign of $T'(\alpha)$. Implicit differentiation of (53) for given $T > 0$ shows that

$$p_T(i;T) = -\frac{g_T(i) - \varphi_T(i)}{g_p(i) - \varphi_p(i)} = -\frac{2\alpha p_i(1-\gamma) - \beta\gamma/T}{2\alpha T(2-\gamma) - \beta(\gamma-1)T/p_i} < 0$$

The same type of calculation shows that $p_\alpha(i;T,\alpha) > 0$, which allows us to conclude that $T'(\alpha) > 0$ by the procedure in Section 3.4. To prove the claimed OR, we show that $R > 0$ uniformly using (9). Thus, we need to calculate the sign of $A(i) - A(j)$ for any $j \rhd i$. As

$$\frac{g_\alpha(i)}{g(i)} = \frac{2p_i T}{2\alpha p_i T + \beta}, \qquad \frac{g_T(i)}{g(i)} = \frac{2\alpha p_i}{2\alpha p_i T + \beta}$$

it is easily verified that $A(i) - A(j) > 0$ proving that $R > 0$ uniformly.

We now show that $R < 0$ uniformly for $d\gamma > 0$. We first show that $T'(\gamma) < 0$. We have

$$p_\gamma(i;T) = -\frac{g_\gamma(i) - \varphi_\gamma(i)}{g_p(i) - \varphi_p(i)} = \frac{c_i p_i^{\gamma-1} T^{\gamma}\left(1 + \gamma Ln(p_i) + \gamma Ln(T)\right)}{2\alpha T(2-\gamma) - \beta(\gamma-1)T/p_i},$$

and thus $p_\gamma(i;T) < 0$ if $1 + \gamma Ln(p_i) + \gamma Ln(T) < 0$, where the last inequality holds as $p_i = t_i/T$ and $t_i > 1$ by presumption. Together with $p_T(i;T) < 0$ this shows that $T'(\gamma) < 0$ by the procedure in Section 3.4. To prove that $R < 0$, we need to verify that $A(i) - A(j) < 0$. But as $\frac{g_\gamma(i)}{g(i)} = 1/\gamma + Ln(p_i) + Ln(T)$, both $\frac{g_\gamma(i)}{g(i)}$ and $\frac{g_T(i)}{g(i)}$ are strictly increasing over agent types, $R < 0$ follows from $T'(\gamma) < 0$, which proves the second claim. ■

Proof of Proposition 18 The equilibrium condition (4) is

$$g(i) \equiv V_1 + (1 - 2p_i)V_2 = c_i p_i^{\eta-1} T^{\eta} \equiv \varphi(i), \tag{54}$$

from which the first claim immediately follows as costs are neutral and V_1 is a level variable if $V_2 = 0$. Hence, let $V_2 > 0$. By Proposition B.2 (Appendix B.4.2) the marginal change

dV_1, dV_2 causes an OR (IR) of $p(\cdot)$ if

$$z(p) \equiv \frac{dV_1 + (1-2p)dV_2}{V_1 + (1-2p)V_2}$$

verifies $z'(p) > (<) 0$. The second claim follows from $sign(z'(p)) = sign\,(dV_1V_2 - dV_2V_1)$. The claim about w_i follows from observing that $dw_i = 2dp_i(1-2p_i)$, showing that w_i must behave like p_i (as $sign\,dw_i = sign\,dp_i$) whenever $p_i < 1/2\ \forall i$. The remaining claim follows from noting that $R_{ij} = 0$ uniformly if $z'(p) = 0$. ■

B Supplementary Material (for Online Publication)

The purpose of this supplementary material is to provide additional results intended to strengthen, elaborate and further generalize our inequality analysis.

Section B.1 proves the equivalence between the equilibria where agents compete directly for market shares or indirectly via choosing their actions. Section B.2 establishes the existence of a unique equilibrium given Assumption 1. Section B.3 elaborates on the continuum representation for atomistic agents. Section B.4 contains additional technical results on rotations. Section B.5 demonstrates how to adopt the formal analysis to study inequality in the Nash equilibrium of aggregative games.

B.1 Equivalence of equilibria

We show that the model, where the agents optimize (1) by choosing $t(i)$ yields the same equilibria as the model, where agents optimize (3) by directly choosing their market shares. For simplicity, we refer to the model with payoff (1) as the "original model", and to the model with payoff (3) as the "transformed model". Recall that an equilibrium of the original model is given by (t, T), where t is the equilibrium action profile, such that $t(i)$ is the optimal action for each $i \in I$, and $T = Z(t)$. The following theorem shows that under the assumptions made, any equilibrium (t, T) of the original model corresponds to an equilibrium (p, T) of the transformed model, and vice-versa.

Theorem B.1 *Suppose that* (2) *holds, and the market share function $p(i, \cdot, T)$ is bijective for any given $i \in I$ and any $T \in \mathbb{R}_+$.*

1) *If (t, T) is an equilibrium of the original model and a function $p : I \to \mathbb{R}_+$ is defined by $p(i) = p(i, t(i), T)$, then (p, T) is an equilibrium in the transformed model.*

2) *If (p, T) is an equilibrium of the transformed model and $t(i) = p^{-1}(i, p(i), T)$ for each $i \in I$, then (t, T) is an equilibrium in the original model.*

<u>Proof</u>: Recall the notational convention that we write $\Pi\left(i, p^{-1}(i, p(i), T), T\right) \equiv \Pi\left(i, p(i), T\right)$ for the payoff function where $t(i)$ has been replaced by the unique corresponding value $p(i)$, instead of using a separate symbolic notation $\Pi, \hat{\Pi}$ to explicitly distinguish between $\Pi(i, t(i), T)$ and $\hat{\Pi}(i, p(i), T)$. In particular, this means that $\Pi(i, t(i), T) = \Pi(i, p(i), T)$ whenever $p(i) = p(i, t(i), T)$ or likewise $t(i) = p^{-1}(i, p(i), T)$.

1) If (t, T) is an equilibrium, then $T = Z(t)$ and for each $i \in I$, $t(i)$ maximizes (1). Let $p(i) \equiv p(i, t(i), T)$ denote the market shares induced by (t, T). Suppose now that (p, T)

as constructed in the Theorem is not an equilibrium of the transformed model. Because $\int_0^1 p(i)di = 1$ by (2), there then must be $\tilde{p}(i) \neq p(i)$ such that $\Pi(i, \tilde{p}(i), T) > \Pi(i, p(i), T)$ for some $i \in I$. Because $\tilde{t}(i) = p^{-1}(i, \tilde{p}(i), T) \neq t(i) = p^{-1}(i, p(i), T)$ this likewise implies that $\Pi(i, \tilde{t}(i), T) > \Pi(i, t(i), T)$ in the original model, contradicting the optimality of t for all $i \in I$.

2) Let (p, T) be an equilibrium of the transformed model, with market shares $p(i) \in \mathbb{R}_+$. Consider the action profile t defined by $t(i) = p^{-1}(i, p(i), T)$, $i \in I$. Because market shares $p(i)$ inversely are determined by $p(i) = p(i, t(i), T)$ and $1 = \int_0^1 p(i)di = \int_0^1 p(i, t(i), T)di$, also $Z(t) = T$ by (2). Now, suppose that (t, T) is not an equilibrium of the original model. Then, there is $\tilde{t}(i) \neq t(i)$ such that $\Pi(i, \tilde{t}(i), T) > \Pi(i, t(i), T)$ for some $i \in I$. Setting $\tilde{p}(i) \equiv p(i, \tilde{t}(i), T)$ for these $i \in I$ then implies that $\Pi(i, \tilde{p}(i), T) > \Pi(i, p(i), T)$, contradicting optimality of $p(i)$ in the transformed model. ■

B.2 Equilibrium existence and uniqueness

We prove that Assumptions 1-2 ascertain the existence of a unique equilibrium $(p(i), T)$.

Theorem B.2 (Existence and uniqueness) *Any model with payoffs* (3) *that satisfy Assumptions 1 and 2 has a unique equilibrium* $(p(i), T)$*. All equilibrium payoffs* $\Pi(i)$ *are positive, and* $p(\cdot)$ *is a bounded, decreasing and strictly positive density.*

The proof evolves in two steps, reflecting the two requirements in the equilibrium definition. Its baseline reasoning is illustrated in Figure 1 in the main text. (A1) implies that a unique optimizer $p(i;T) > 0$ exists for any given $T > 0$ and any fixed $i \in [0,1]$. This follows because a zero market share is not optimal ($g(i, 0, T) > 0 = \varphi(i, 0, T)$), the gains from increasing one's market share are limited ($g(\cdot, T)$ bounded from above) and marginal costs are strictly increasing in $p(i)$ and unbounded. Uniqueness of this optimizer is implied by strong quasiconcavity. (A2) then assures that there is a unique $T > 0$ such that $\int p(i;T)di = 1$. To see why, suppose that $g(i, p, T)$ is bounded above and away from 0 for any $p \geq 0$ and any $T > 0$, consistent with (but stronger than) assumption (A2). Even the best agent ($i = 0$) then seeks to make her market share $p(i;T)$ arbitrarily small because as $T \to \infty$ marginal costs become arbitrarily large. Similarly, even the worst agent ($i = 1$) aims at an arbitrarily large $p(i;T)$ if $T \to 0$ and marginal costs become arbitrarily small. These two facts imply that $\lim_{T\to\infty} \int p(i;T)di = 0$ and $\lim_{T\to 0} \int p(i;T)di = \infty$, and existence of a $T > 0$ with $\int p(i, T)di = 1$ follows by continuity of $\int p(i; \cdot)di$. Uniqueness then follows from the last assumption in (A2), which assures that $\int p(i; \cdot)di$ is strictly decreasing at $\int p(i, T)di = 1$.

Proof of Theorem B.2 The proof consists of two steps. i) Fix $i \in [0,1]$ and $T > 0$ arbitrarily. (A1) assures that the equation (4) has a unique solution $p(i;T) > 0$, and that this solution indeed maximizes (3) given T. Now, consider the function $p(i,T) \equiv p(i;T)$, noting that $p(i,\cdot)$ is a strictly decreasing C^1-function on $(0,\infty)$ as a consequence of the Implicit Function Theorem, the strong quasiconcavity assumption in (A1), and the last assumption of (A1). Moreover, $p(\cdot,T)$ must be a decreasing function by Assumption 2 and, hence, $p(\cdot,T)$ is integrable over $[0,1]$, so let $G(T) \equiv \int_0^1 p(i,T)di$, noting that G is differentiable. ii) We show: $\exists!\ T > 0$: $G(T) = 1$. Fix $i \in [0,1]$. By (A2) there must exist $T_i > 0$: $g(i,1,T_i) = \varphi(i,1,T_i)$. Therefore, $\exists T_0 > 0$ such that $p(0,T_0) = 1$. Because $p(i,\cdot)$ strictly decreasing, it follows that $p(0;T) < 1$ for $T > T_0$. Since $p(\cdot,T)$ is decreasing, we must have $p(i,T) < 1$ for any $i \in [0,1]$ and $T > T_0$, which implies that $\lim_{T\to\infty} G(T) < 1$. Similarly, it follows that $\exists T_1 > 0$ such that $p(1;T_1) = 1$. Thus $p(i,T_1) > 1$ for $i \in [0,1]$ and $T < T_1$, hence $\lim_{T\to 0} G(T) > 1$. As $G(\cdot)$ continuous, $\exists T > 0$ such that $G(T) = 1$, and uniqueness follows from the fact that, for each $i \in [0,1]$, $p(i;T)$ and hence $G(T)$ is strictly decreasing in T. Finally, $\Pi(i) > 0$, because $p(i) = p(i;T) > 0$ is the unique maximizer and $\Pi(i)|_{p(i)=0} = 0$. ■

B.3 Continuum representation for atomistic agents

We illustrate that the equilibrium distribution in case of $n \in \mathbb{N}$ atomistic ("discrete") agents can be identified with our finite step density model. The following argument considers the case, where heterogeneity enters the model through a cost coefficient function as in (5). This should suffice to make evident that the representation result applies similarly to other cases as well. Consider a population consisting of $n \in \mathbb{N}$ atomistic (or "discrete") agents, indexed by $\{1/n, 2/n, ..., 1\}$. Suppose that the agents differ in their cost coefficient $c(i)$, $i \in \{1/n, 2/n, ..., 1\}$. Then, the agents can be partitioned into $1 \leq K \leq n$ groups of identical agents, with group size n_k, $\sum_k n_k = n$. This partition gives $1 \leq K \leq n$ equivalence classes (groups) of sizes $n_1, ..., n_K$, $\sum_k n_k = n$. We identify each group by a "representative" agent i_k. In equilibrium every agent (i/n) chooses $p^d(i/n)$ (d for "discrete") to maximize her payoff $\Pi(i)$, where $p^d(i/n)$ must satisfy $\sum_{i=1}^n p^d(i/n) = 1$. Let $p(i)$ denote the (step) density function that characterizes our (continuum) equilibrium from definition 1 with the corresponding cost step function $c(i) = c(i_k)$ on $[i_k, i_{k+1})$, and group measures $\gamma_1, ..., \gamma_K$ satisfying $\gamma_k = n_k/n$. We now establish the formal equivalence between the discrete equilibrium probability distribution $\{p^d(1/n), ..., p^d(1)\}$ and the equilibrium step density $p(i)$.

Theorem B.3 (Continuum Representation) *Let $n \in \mathbb{N}$ and suppose that agents are partitioned in K cost groups. If $\{p^d(i/n)\}$ corresponds to the discrete equilibrium and $p(i)$ is the equilibrium (step) density of the respective continuum problem, then $p^d(i/n) = \frac{1}{n}p(i/n)$ is satisfied for all $i \in \{1, ..., n\}$*

Proof: In the continuum case we only have to solve the optimization problem for a representative agents i_k. In the discrete equilibrium $1 = \sum_{i=1}^{n} p^d(i/n) = \sum_{k=1}^{K} p^d(i_k)n_k$. The claim now is that $\frac{1}{n}p(i_k) = p^d(i_k)$ for $k = 1, ..., K$. But because in the continuum equilibrium we must have

$$1 = \int_0^1 p(i)di = \sum\nolimits_{k=1}^{K} p(i_k)\gamma_k = \sum\nolimits_{k=1}^{K} \left(\frac{1}{n}p(i_k)\right) n_k$$

the claim follows from the uniqueness of equilibrium. ■

Hence the continuum step-function case and the atomistic case are equivalent up to the multiplicative constant $1/n$ (independent of group composition), which means that we can work with either model, and justifies our procedure of the main text. It then also follows that $p(i_k)\gamma_k = p^d(i_k)n_k$ corresponds to the market share of a member of group k, illustrating why we used the notion of a "representative" agent.

Because Theorem B.3 remains valid as n grows arbitrarily large, this provides the following justification for using strictly increasing cost coefficient functions (Class II) as an approximation for the case of many different agents. Suppose that $c(i)$ is a Class II function defined on $[0, 1]$ (e.g., $c(i) = 1+i$), and let $p(i) = p(c(i))$ denote the corresponding equilibrium density (a strictly decreasing, continuous function). Then, because $c(i)$ is continuous on a compact interval, for $n \in \mathbb{N}$ the sequence of step functions defined by $c_n(i) = c(i)$ if $i \in \{0, 1/n, 2/n, ..., 1\}$ and $c_n(i) = c(s/n)$ for $i \in (s/n, (s + 1)/n)$, $s \in \{0, 1, ..., n - 1\}$, converges (uniformly) to $c(i)$ as $n \to \infty$.[47] Consider the atomistic equilibrium distribution $p^d(i/n)$ induced by $c(0), c(1/n), c(2/n), ..., c(1)$. By Theorem B.3, $np^d(i/n) = p(i/n)$, where $p(i/n)$ is the step-density version of $p^d(i/n)$. More precisely, for a given $n \in \mathbb{N}$ this density is a decreasing finite step function with $p_n(i) = p(c_n(i))$, where $c_n(i)$ is as defined above. Because $c_n(i) \to c(i)$ and $p(i)$ is continuous, we have $p(i) = p(c(i)) = p(\lim c_n(i)) = \lim p(c_n(i)) = \lim p_n(i)$. This shows that while, of course, the atomistic $p^d(i/n)$ becomes arbitrarily close to zero as n grows large, the "scaled" distribution law as captured by the step-density version $p(i/n)$ approaches $p(i)$.

[47] Such approximations of continuous functions by a sequence of step functions are a standard result in real analysis and integration theory.

B.4 Rotations: Further Results

B.4.1 Ratio Test

We prove that condition (11) from Section 3.3.2 indeed implies a rotation of $p(\cdot)$.

Proposition B.1 *Suppose that $\infty > p(\cdot; x'), p(\cdot; x) > 0$ are right-continuous, decreasing SSD densities with similar equivalence classes $[i]$. If (11) holds, then $p(\cdot; x')$ is an OR (IR) of $p(\cdot; x)$.*

Proof: Let $g(i) \equiv \frac{p(i;x')}{p(i;x)}$. Because $p(\cdot; x')$ and $p(\cdot; x)$ both are SSD densities with similar equivalence classes, (11) implies that $g(\cdot)$ is not constant on $(0, 1)$: $\exists i_0 \in (0, 1)$ such that $g(i) > g(j)$, $i < i_0 \leq j$. Further, $g(\cdot)$ is right-continuous, and by (11) also decreasing. We first claim that $g(0) > 1$. Suppose, by contradiction, that $g(0) \leq 1$. Then $g(i) \leq 1$, $i \in I$, and additionally $g(j) < 1$ for $j \geq i_0$. This implies that $p(i; x') \leq p(i; x)$, $i \in I$, and $p(j; x') < p(j; x)$ for $j \geq i_0$. Hence $\int p(s; x')ds < \int p(s; x)ds$, contradiction. Therefore $g(0) > 1$; a similar argument shows that $g(1) < 1$. Because $g(0) > 1$ and $g(\cdot)$ is right-continuous, the set $\{i : g(i) > 1, i > 0\}$ is non-empty, and we let $i_0 \equiv sup\{i > 0 : g(i) > 1\}$, noting that $i_0 \in (0, 1)$. Because g decreases and $g(1) < 1$, the set $\{i : g(i) < 1, i \geq i_0\}$ is non-empty, and we set $i_1 \equiv inf\{i \geq i_0 : g(i) < 1\}$. If $i_0 = i_1$, then $p(i; x') > p(i; x)$ on $(0, i_0)$, and $p(i; x') < p(i; x) > 0$ on $(i_0, 1]$. If $i_0 < i_1$ then $g(i) = 1$ on (i_0, i_1). These facts together imply that $p(\cdot; x')$ is OR of $p(\cdot; x)$; the case on an IR is proved similarly. ■

B.4.2 On the Direct-Aggregative Effect

By Theorems 1 and 2, the sign of the direct-aggregative effect is crucial for studying the inequality effects. In this section we search for the determinants of $sign(R_{ij})$ in terms of primitives. We concentrate on the multiplicatively separate case where $\varphi(i) = \varphi(i, p)C(T)$ as many of our applications feature such a cost function.

Proposition B.2 *Let $\varphi(i) = \varphi(i, p)C(T)$, and define $h(i, p, T; x) = Ln\left(g(i, p, T; x)\right)$. If*

$$h_x(i, p, T; x_0) \geq h_x(j, p, T; x_0), \quad h_T(i, p, T; x_0) \geq h_T(j, p, T; x_0) \quad \forall j \rhd i,$$
$$h_T(i, p', T; x_0) \geq h_T(i, p, T; x_0), \quad h_x(i, p', T; x_0) \geq h_x(i, p, T; x_0) \quad \forall i \text{ and any } p' > p,$$

where at least one of the above inequalities is strict, then R is uniformly positive at x_0. If all inequalities are reversed (and one strict so), then R is uniformly negative at x_0. Finally, if all inequalities are equalities, then R is uniformly zero at x_0.

Proof: We only prove the uniformly positive case (the negative case is established by the same type of arguments). We need to show that for

$$A(i) = \frac{g_T(i)}{g(i)}T'(x) + \frac{g_x(i)}{g(i)}$$

we have $A(i) > A(j)$ whenever $j \rhd i$. So take any $j \rhd i$. First, $h_T(i,p',T;x_0) \geq h_T(i,p,T;x_0)$ and $h_T(i,p,T;x_0) \geq h_T(j,p,T;x_0)$ yield

$$h_T(i,p(i),T;x_0) \geq h_T(i,p(j),T;x_0) \geq h_T(j,p(j),T;x_0)$$

and because $T'(x) > 0$ (see Lemma 3, Section 3.4) and $h_T = \frac{g_T}{g}$ we have

$$\frac{g_T(i)}{g(i)}T'(x) \geq \frac{g_T(j)}{g(j)}T'(x),$$

where the inequality is strict, whenever at least one of the initial inequalities is strict. Second, $h_x(i,p,T;x_0) \geq h_x(j,p,T;x_0)$ and $h_x(i,p',T;x_0) \geq h_x(i,p,T;x_0)$ yield

$$h_x(j,p(j),T;x_0) \leq h_x(j,p(i),T;x_0) \leq h_x(i,p(i),T;x_0)$$

and hence also

$$\frac{g_x(i)}{g(i)} \geq \frac{g_x(j)}{g(j)}$$

where, again, the inequality is strict if one of the previous inequalities is strict. This shows that $R_{ij}(x_0) > 0$. Finally, is all inequalities are equalities, the condition in Proposition B.2 is equivalent to multiplicative separability of $g(i)$ in (i,p) and (T,x), meaning that x must be a level variable, and the last claim follows from Corollary 3. ■

Proposition B.2 contains the inequality analysis of the contest model from section 5.2 as the special case, where marginal benefits in (4) are independent from p and T. In such a case, Proposition B.2 tells us that only the direct effect of dx matters for the inequality effects. In particular, incentives to increase market shares are relatively stronger for strong agent types iff marginal benefits increase proportionally more for these types, i.e., iff $dg(i)/g(i) > dg(j)/g(j)$ holds for $j \rhd i$. In the more general case, where marginal benefits depend also on p and T, the above effect is mitigated. For example, a positive shock $dx > 0$ increases T by Lemma 3, and the direct incentive effects of $dx > 0$ are either reinforced or weakened in response to dT depending on g_T.[48]

As another application, suppose that marginal benefits can be written as a power function of the form

$$g(i, p, T, x) = u(i)v(T; x)p^{\xi(T;x)},$$

Proposition B.2 then implies that R is determined solely by the elasticity function $\xi(T, x)$. In particular, R is uniformly positive if $\xi_T, \xi_x \geq (\leq)0$ with one strict inequality, and $R = 0$ uniformly if ξ is constant.

B.5 Market Inequality in the Nash Equilibria of Aggregative Games

The aim of this section is to exemplify, by means of a simple application from contest theory, that the approach of this paper can be applied to study inequality effects in aggregative games with payoffs of the form (1). The only essential difference to the setting of the main text is that in an aggregative game the individual agents take into account the effects of their own actions on the aggregate in a Nash equilibrium. If the aggregate T in payoff (1) is not exogenous to the individual player, it is not obvious whether and how our distributional tools may be used to study the inequality effects.

We now exemplify how our approach can be adjusted to be fruitful also in this more cumbersome case. The way how we proceed to use our inequality tools if individual players take into account their impact on the aggregate is general, and can be used in any other aggregative game with a differentiable structure as well. Note that there is little loss in assuming a *sum*-aggregative structure of such games, because any aggregative game with a well-behaved aggregator (differentiable and coordinate-wise strictly monotonic) is strategically equivalent to a sum-aggregative game (Cornes and Hartley, 2012).

In Section 5.2 we analyzed a contest model under the assumption that individual agents take the aggregate $T = \int t(i)di$ as given when choosing their effort, while T was endogenous to the model. We now study the same model, assuming that each of finitely many agent types takes its own effect on the aggregate into consideration. Consider a fixed-prize contest with n atomistic agents and payoffs

$$\Pi_i = \frac{t_i}{\sum t_s} V - \Phi(i, t_i) \tag{55}$$

[48] Particularly, if $g_T > (<)0$, then $R > 0$ is more likely to result if the increase (decrease) in marginal benefits triggered by $dT > 0$ affects the strong agent relatively more (less).

Define the market share $p_i = t_i/T$ as before, with $T \equiv \sum t_s$. Let $T_i \equiv \sum_{s \neq i} t_s$. Because $T = T_i + t_i$ and $t_i = p_i T$ we obtain $t(i) = \frac{p_i}{1-p_i} T_i$. Thus we can restate (55) in terms of own market share as

$$\Pi_i = p_i V - \Phi\left(i, \frac{p_i}{1-p_i} T_i\right), \tag{55'}$$

where T_i is exogenous to player i. A Nash equilibrium is a probability vector $(p_1, ..., p_n)$ and an aggregate $T > 0$ such that $T_i = (1 - p_i)T$ and $p(i)$ maximizes (55').[49] Thus, any interior Nash equilibrium satisfies the FOC system

$$V = \frac{\varphi(i, p_i T)}{1 - p_i} T. \tag{56}$$

Because (56) is of the form (4), we can apply the inequality tools from Section 3 as they are to study the inequality effects as we did in the main text.[50]

One question of self-interest is whether the inequality predictions are sensitive to the change in the equilibrium concept. For example, Acemoglu and Jensen (2010) find that sometimes individual strategies may respond differently to exogenous shocks if players take their impact on the aggregate into account. In this respect, the following proposition shows that we find the *same* inequality effects induced by an increase in the common prize value $dV > 0$ in case of Nash equilibria, at least in the present contest model.

Proposition B.3 *The inequality effects induced by dV are determined by the t-elasticity of marginal costs alone. If $\Phi(i,t) = c_i t^\gamma$, then $p(\cdot)$ is invariant to x. If $\Phi(i,t) = ct^{\gamma_i}$ such that p_i is of Class I, then $dV > 0$ causes an OR of $p(\cdot)$.*

Proof: Evaluating (9) in case of (56) and using $t_i = p_i T$ yields

$$sign(R_{ij}) = sign\left(\frac{\varphi_t(j,t_j)t_j}{\varphi(j,t_j)} - \frac{\varphi_t(i,t_i)t_i}{\varphi(i,t_i)}\right)$$

proving the first claim. With iso-elastic costs we obtain $\varphi(i, p_i T) = \gamma c_i {p_i}^{\gamma-1} T^{\gamma-1}$. By (56), this implies that

$$\frac{c_j}{c_i} = \left(\frac{p_i}{p_j}\right)^{\gamma-1} \left(\frac{1-p_j}{1-p_i}\right)^{\gamma+1},$$

from which $R_{ij} = 0$ follows. A similar argument together with the fact that $T'(V) > 0$ shows

[49] It is straightforward to verify by the same arguments we use in Theorem B.3 that this characterization of Nash equilibrium is equivalent to the standard one.

[50] A formal advantage of the model in the main text is that it yields a more tractable structure. The respective FOC is $V = \varphi(i, p_i T)T$. For given p_i, T marginal costs are thus higher if the own effect on the aggregate are taken into account. This is intuitive, because an increase in $t(i)$ also increases T which, ceteris paribus, decreases t_i/T.

that $R_{ij} > 0$ if Φ is iso-elastic with exponents γ_i that increase over agent types, which shows the OR of $p(\cdot)$. ■